National Centre for Nuclear Research

Doctoral Thesis

Expanding the accessible kinematic domain of generalized parton distributions

Author:
Víctor Martínez-Fernández

Supervisor:
Jakub Wagner
Auxiliary supervisor:
Paweł Sznajder

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

Department of Theoretical Physics

NATIONAL CENTRE FOR NUCLEAR RESEARCH ŚWIERK

June 11, 2024

Declaration of Authorship

I, Víctor MARTÍNEZ-FERNÁNDEZ, declare that this thesis titled "EXPANDING THE ACCESSIBLE KINEMATIC DOMAIN OF GENERALIZED PARTON DISTRIBUTIONS" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at the National Centre for Nuclear Research.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at the National Centre for Nuclear Research or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ________________________________

Date: ________________________________

“There remains the question: what then has to replace the concept of a fundamental particle? I think we have to replace this concept by the concept of a fundamental symmetry. The fundamental symmetries define the underlying law which determines the spectrum of elementary particles. [...] I only wanted to say that what we have to look for are not fundamental particles but fundamental symmetries.”

W. Heisenberg,
“The Physicist’s conception of Nature” (1973).

NATIONAL CENTRE FOR NUCLEAR RESEARCH

Abstract

EXPANDING THE ACCESSIBLE KINEMATIC DOMAIN OF GENERALIZED PARTON DISTRIBUTIONS

Víctor MARTÍNEZ-FERNÁNDEZ

Quantum Chromodynamics (QCD) is the theoretical framework to study hadrons by means of their fundamental degrees of freedom, i.e. quarks and gluons, collectively referred to as partons. QCD defines many types of distributions describing a given nucleon in terms of partons. In this doctoral dissertation, we are interested in the so-called generalized parton distributions (GPDs). GPDs are off-forward matrix elements of quark and gluon operators that serve as a window to the total angular momentum of partons and their transverse imaging (nucleon tomography).

GPDs arise from the factorization that takes place in scattering amplitudes when a particle probing the hadron, typically a virtual photon, is considered in the limit of infinite virtuality. This is known as the kinematic leading twist (LT). GPDs are accessed in exclusive process where the states of all incoming and outgoing particles are measured. Three such processes, which are prominent for current and near-future experimental programmes, are: deeply virtual Compton scattering (DVCS), timelike Compton scattering (TCS) and double deeply virtual Compton scattering (DDVCS). Current and future data can not be considered as a practical realization of the aforementioned LT approximation, suggesting the need for corrections inversely proportional to the photon virtuality. In QCD, these ones are referred to as kinematic and genuine higher-twist corrections.

At the lowest approximation, DVCS and TCS grant access to GPDs in a particular region of the partonic kinematics. Such a limitation is not present in DDVCS, which serves as the main motivation for this doctoral project. Since DVCS and TCS are special limits of DDVCS, this project also provides a consistent framework for the description of the three processes. Therefore, as a first task, we consider DDVCS at leading order (LO) in the strong coupling constant and LT, and address the feasibility of measuring it at current and future experiments. As indicated above, since these experiments do not fulfill the conditions for a correct LT description, our next task is to calculate the kinematic twist corrections for DDVCS at LO. From the formulation of DDVCS, we can obtain the corresponding corrections for DVCS and TCS and finally provide numerical estimates of these effects, leaving aside the genuine higher-twist corrections which are a difficult and separate subject.

NATIONAL CENTRE FOR NUCLEAR RESEARCH

Streszczenie

Expanding the accessible kinematic domain of generalized parton distributions

Víctor Martínez-Fernández

Chromodynamika kwantowa (ang. *quantum chromodynamics*, QCD) jest teorią opisującą strukturę hadronów za pomocą kwarków i gluonów, które łącznie określane są mianem partonów. QCD definiuje różne typy rozkładów opisujących dany hadron. Niniejsza praca doktorska dotyczy tzw. uogólnionych rozkładów partonów (ang. *generalised parton distributions*, GPDs), które są niediagonalnymi elementami macierzowymi operatorów kwarków i gluonów. Rozkłady GPD służą m.in. do badania całkowitego momentu pędu partonów oraz obrazowania ich na płaszczyźnie poprzecznej do ruchu hadronu (tzw. tomografia nukleonowa).

Rozkłady GPD pojawiają się jako elementy faktoryzacji amplitud procesów ekskluzywnych, w idealnym przypadku, gdy cząstka sondująca hadron, zazwyczaj wirtualny foton, ma nieskończoną wirtualność (tzw. przybliżenie wiodącego twist-u, ang. *leading twist approximation*). Przykładami procesów ekskluzywnych, czyli takich, w których stany wszystkich cząstek biorących udział w oddziaływaniu są mierzone, są: głęboko-wirtualne rozpraszanie comptonowskie (ang. *deeply virtual Compton scattering*, DVCS), czasopodobne rozpraszanie comptonowskie (ang. *timelike Compton scattering*, TCS) i podwójne głęboko-wirtualne rozpraszanie comptonowskie (ang. *double deeply virtual Compton scattering*, DDVCS). Wszystkie te procesy są ważne z punktu widzenia obecnych i przyszłych programów eksperymentalnych związanych z rozkładami GPD. Dane z nimi związane nie mogą jednak być uważane za praktyczną realizację przybliżenia wiodącego twist-u, co sugeruje konieczność wprowadzenia poprawek do opisu procesów ekskluzywnych odwrotnie proporcjonalnych do wirtualności fotonu. W QCD poprawki tego typu znane są pod nazwą kinematycznych oraz właściwych (ang. genuine) poprawek wyższych twist-ów (ang. *higher twist corrections*).

Podstawowy opis procesów DVCS i TCS daje dostęp do rozkładów GPD jedynie w ściśle określonej kinematyce partonów. Ograniczenie to nie występuje w przypadku procesu DDVCS, co stanowi główną motywację dla niniejszej pracy doktorskiej. W pierwszej części analizy opisujemy DDVCS na poziomie wiodącym w stałej sprzężenia silnego (ang. *leading order*, LO) i w przybliżeniu wiodącego twist-u, dodatkowo analizując możliwość pomiaru tego procesu w obecnych i przyszłych eksperymentach. Kolejnym elementem analizy jest wyznaczenie kinematycznych poprawek wyższych twist-ów dla DDVCS na poziomie LO, z uwzględnieniem numerycznego oszacowania wpływu tych poprawek. Ponieważ DVCS i TCS są specjalnymi przypadkami DDVCS, uzyskane wyniki można użyć również dla tych procesów, dzięki czemu praca ta dostarcza spójny opis wszystkich trzech wymienionych procesów ekskluzywnych (pomijając „właściwe" wyższe twisty, które są trudnym i odrębnym zagadnieniem).

Acknowledgements

Firstly, I would like to thank my supervisors, Jakub Wagner and Paweł Sznajder, for their support, patience and respect. They have always listened to my ideas and discussed them as peers, kindly correcting my mistakes and celebrating my successes. Also, I could not forget to thank Bernard Pire who has been working with us all the way during these (almost) four years and offered to proofread this thesis. I had the opportunity and pleasure to work side by side with him at the École Polytechnique where we struggled computing higher twists, but eventually accomplished it.

Big thanks are in order to Vladimir Braun and Alexander Manashov who invited me to the University of Regensburg, where I was able to learn their techniques for the conformal operator-product expansion. They helped me to understand better conformal field theory and answered all my inquires. I greatly appreciate their help since these techniques constitute the grounds for chapters 3 and 4 of this dissertation.

Furthermore, I would like to acknowledge the financial support provided by NAWA (Narodowa Agencja Wymiany Akademickiej[1]), and NCN (Narodowe Centrum Nauki[2]) under PRELUDIUM grant 2021/41/N/ST2/00310.

Last but not least, I am truly grateful to my family and friends for always being *there* so that I could be *here*.

[1]In English, Polish National Agency for Academic Exchange.
[2]In English, National Science Centre.

Contents

To my parents

1

Introduction

We can consider that the modern approach to particle physics through symmetries began with the introduction of the *isospin* quantum number, I, by W. Heisengberg in 1932 [1]. He proposed to classify the proton and neutron as two different states of the same particle, the *nucleon*, differentiated by their isospin values: $I = +1/2$ for the proton and $I = -1/2$ for the neutron. This way, the nucleon belongs to the fundamental representation of an approximate $SU(2)_I$ symmetry. It is approximate because the neutron is slightly heavier than the proton.

The discovery of new particles, later called *hadrons*, led to the introduction of more quantum numbers: strangeness, charmness, etc. In an attempt to unveil the relationship between these new charges and particles, M. Gell-Mann [2] and G. Zweig [3] proposed independently the existence of some elemental particles, today known as *quarks*, that would serve as building blocks for the hadrons. These particles would belong to the fundamental representation of a $SU(n)_f$ symmetry and they would be differentiated by their *flavor* quantum number, in correspondence with the aforementioned isospin. For $n \leq 5$, one finds hadrons made out of quarks with flavors u, d, c, s and b. There is a sixth flavor, t, but this kind of quark is too heavy to hadronize, decaying via weak force to a b quark: $t \to b + W^+$. The flavor symmetry is broken as the different quarks have different masses. Under the quark model, both nucleon states can be understood as made out of three valence quarks (carrying the quantum numbers of the hadron): $n \sim udd$ and $p \sim uud$. The difference in the quark content, together with electrodynamic effects [4], is what elevates the neutron with respect to the proton in the mass spectrum.

With the quark model at hand, one can organize the different hadrons into baryons (with three valence/constituent quarks) and mesons (with a quark-antiquark pair), which in turn can be arranged in multiplets following the irreducible representations of the $SU(n)_f$ flavor group. For $n = 3$ (quarks u, d and s) we have the Clebsch-Gordan decomposition of baryons:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}\,, \tag{1.1}$$

and of mesons:

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}\,. \tag{1.2}$$

Because quarks (antiquarks) belong to the fundamental (antifundamental) representation $\mathbf{3}$ ($\bar{\mathbf{3}}$) of the $SU(3)_f$ group, they are the fundamental degrees of freedom of the theory and hence the elemental particles. They are the building blocks of hadrons belonging to three irreducible representations: **10** (decouplet), **8** (octet) and **1** (singlet).

The quark model gained support after the discovery of the particle $\Omega^- \sim sss$ in 1964 [5], formerly predicted by Gell-Mann. Nevertheless, this model suggests that the Ω^- particle consists of three identical quarks. These ones carry flavor and spin which can only take two values ($\pm 1/2$), not enough to build an antisymmetric 3-quark ground state. This means that in order to satisfy the Pauli exclusion principle, a third quantum number must be introduced. For this purpose, O. W. Greenberg [6] proposed the existence of a new charge, the *color*, that could take three values: red (R), green (G) and blue (B). Color is related to an exact $SU(3)_c$ symmetry so that, in this case, quarks with different color but the same flavor are equally massive.

The Clebsch-Gordan decomposition for three colors corresponds to that of Eq. (1.1). Since this charge is not observable, all particles are supposed to be realizations of the singlet representation for which the color charges of the different quarks cancel: the "total color" is white. The $SU(3)_c$ color group has eight generators which translate to eight different bosonic carriers of the color interaction called *gluons*. The existence of these particles was confirmed in hadronic jet production from electron-positron annihilations. A jet consists of a collection of hadrons that are produced by an initial parton whose color field stimulates the creation of quark-antiquark pairs that rearrange forming mesons and baryons. At the parton level, the electron-positron annihilation may produce in the simplest cases a quark-antiquark pair ($\mathfrak{q}\bar{\mathfrak{q}}$) or a quark-antiquark-gluon system ($\mathfrak{q}\bar{\mathfrak{q}}A$):

$$e^- + e^+ \to \mathfrak{q} + \bar{\mathfrak{q}}\,, \tag{1.3}$$
$$e^- + e^+ \to \mathfrak{q} + \bar{\mathfrak{q}} + A\,. \tag{1.4}$$

The case (1.3) produces two back-to-back jets and was firstly reported by the Stanford Linear Accelerator Center (SLAC) [7, 8], while the reaction (1.4) generates three jets and was confirmed by the Deutsches Elektronen-Synchrotron (DESY) [9–12]. Since the infrared divergences of the partonic cross-sections for the reactions above cancel out, the large distance angular distributions of the hadronic jets are those of the parent particles, i.e. the quark, antiquark and gluon. Consequently, the detection of three jets is an experimental confirmation of the existence of gluons.

An interesting characteristic of these bosons is that, as opposed to photons, they can interact among themselves. The reason as to why lays on the fact that photons do not have electric charge while gluons do carry color. This feature is realized in the theory through the non-commutativity of the generators of the $SU(3)_c$ group:

$$[T_i, T_j] = i \sum_{k=1}^{8} f_{ijk} T_k\,, \quad f_{ijk} \in \mathbb{R}\,, \quad T_i^\dagger = T_i\ \forall\, i\,, \tag{1.5}$$

where $\{T_i, i = 1, ..., 8\}$ is the set of generators and f_{ijk} are the group structure constants. In other words, $SU(3)_c$ is a non-Abelian group, as opposed to the $U(1)$ group describing the electromagnetic interactions by exchanges of photons. As a result, the field strength tensor for the ith gluon with vector field A_i^μ takes the form

$$F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial^\nu A_i^\mu + 2g_s \sum_{j,k=1}^{8} A_j^\mu A_k^\nu f_{ijk}\,, \tag{1.6}$$

where g_s is the strong coupling constant which is often introduced in cross-sections via the alternative quantity

$$\alpha_s = \frac{g_s^2}{4\pi} . \tag{1.7}$$

The corresponding covariant derivative acting on a quark field of flavor f represented as

$$\mathfrak{q}_f = \begin{pmatrix} \mathfrak{q}_{f,R} \\ \mathfrak{q}_{f,G} \\ \mathfrak{q}_{f,B} \end{pmatrix} \quad \text{with } R, G, B \text{ the color indices,} \tag{1.8}$$

is given by

$$D_\mu \mathfrak{q}_f = \partial_\mu \mathfrak{q}_f - i g_s \sum_{i=1}^{8} T_i A_{i,\mu} \mathfrak{q}_f . \tag{1.9}$$

The theory that describes the interaction between quarks mediated by gluons and, therefore, explains the structure of hadrons is the *quantum chromodynamics* (QCD). The Lagrangian of QCD is gauge-invariant under $SU(3)_c$ transformations and with the objects defined above can be written as

$$\mathscr{L} = \sum_f \bar{\mathfrak{q}}_f (i\not{D} - m_f)\mathfrak{q}_f - \frac{1}{4}\sum_{i=1}^{8} F_{i,\mu\nu}F_i^{\mu\nu} + \mathscr{L}_{\text{ghosts}} + \mathscr{L}_{\text{gauge}} . \tag{1.10}$$

where the parameter m_f is the mass of the quark of flavor f and the covariant derivative D_μ appears multiplied by a Dirac-gamma matrix γ^μ, $\not{D} = \gamma^\mu D_\mu$. The gamma matrix is required to make the quark dynamics invariant under Lorentz transformations. The component $\mathscr{L}_{\text{ghosts}}$ accounts for the Fadeev-Popov ghost fields ($\mathfrak{c}$) [13]

$$\mathscr{L}_{\text{ghosts}} = i \sum_{a,b} \bar{\mathfrak{c}}_a \partial_\mu (D^\mu)_{ab} \mathfrak{c}_b , \tag{1.11}$$

where a, b are indices in the adjoint representation of $SU(3)_c$, and $\mathscr{L}_{\text{gauge}}$ for the choice of gauge for the gluons

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{2\chi}\sum_{i=1}^{8}(\partial_\mu A_i^\mu)^2 . \tag{1.12}$$

The parameter χ can be chosen in different ways being the most popular ones the *Feynman gauge* for which $\chi = 1$, the *Landau gauge* $\chi \to 0$ and the *unitary gauge* $\chi \to \infty$. Observables turn out to be independent of the selected value.

Contrary to the *quantum electrodynamics* (QED), which studies the electromagnetic interaction and is based on the Abelian ($f_{ijk} = 0 \;\forall\, i,j,k$) group $U(1)$, the existence of non-vanishing structure constants for the group $SU(3)_c$ allows gluons to interact among themselves. This feature which was introduced earlier in this chapter is now immediate by inspection of the gluon terms in the QCD Lagrangian (1.10):

$$\sum_{i=1}^{8} F_{i,\mu\nu}F_i^{\mu\nu} \supset -\frac{g_s}{2}\sum_{i,j,k=1}^{8} f_{ijk}A_{j,\mu}A_{k,\nu}\partial^\mu A_i^\nu + 4g_s^2 \sum_{i,j,k,J,K=1}^{8} f_{ijk}f_{iJK}(A_jA_J)(A_kA_K) , \tag{1.13}$$

which represents the three- and four-gluon vertices and from where we can conclude that there is no free gluons theory. Gluons are always self-interacting.

The interactions among quarks and gluons are responsible for the *running* of the

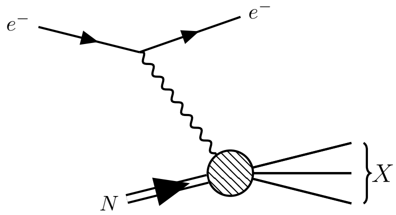

Fig. 1.1. Feynman dagram for deep inelastic scattering (DIS) of nucleon N and electron e^-. The set of undetected particles that come out of the scattering is denoted as X.

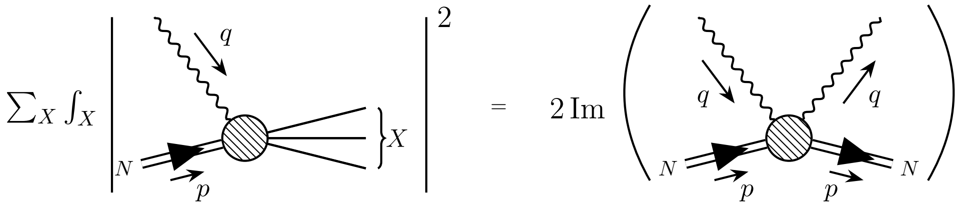

Fig. 1.2. Optical theorem applied to DIS. Here, $\sum_X \int_X$ corresponds to summation over all undetected species X and integration over their momenta as in Eq. (1.19).

QCD coupling g_s. The running is such that the coupling grows as the energy scale gets reduced, making quarks to be confined within the hadrons. This particular feature of QCD is known as *color confinement*. Conversely, the coupling decreases as the energy scale increases, so that quarks behave as free particles in the limit of large scale. This characteristic is referred to as *asymptotic freedom* [14, 15].

In the field of QCD, it is custom to refer to gluons and quarks collectively as *partons*, name proposed by R. P. Feynman [16]. The first experiments addressing the parton content of the proton focused on a particular process named *deep inelastic scattering* (DIS) [17]. This reaction consists of an electron beam scattering off a nucleon, see Fig. 1.1:

$$N(p) + e^-(k) \to e^-(k') + X\,, \tag{1.14}$$

where $N(p)$ represents a nucleon with four-momentum p and X the set of undetected particles. The electron e^- transitions from a state with momentum k to one with momentum k'. The set of states X is not measured in the experiment, and because of that this process is classified as *inclusive*. From the theory side, this can be implemented by an integration over those states which allows us to relate DIS to a process where final and initial states coincide. This is achieved by means of the optical theorem which claims

$$\sum_F |\mathcal{T}_{FI}|^2 = 2\,\mathrm{Im}\{\mathcal{T}_{II}\}\,, \tag{1.15}$$

with $\mathcal{T}_{FI}$ representing the transition matrix from the initial state I to the final one $F \neq I$, provided that the scattering matrix, $\mathcal{S}$, is expanded as

$$\mathcal{S} = 1 + i\mathcal{T}\,. \tag{1.16}$$

The Feynman diagrams in Fig. 1.2 illustrate the application of the optical theorem (1.15) to DIS.

The transition matrix for the reaction (1.14) to leading order (LO) in the strong coupling constant can be written as

$$i\mathcal{T}_{\text{DIS}} = (2\pi)^4 \delta(k + p - k' - p_X) \frac{ie^2 \bar{u}(k', s')\gamma^\mu u(k, s)}{Q^2} \langle X | j_\mu(0) | p \rangle \,, \tag{1.17}$$

where e is the absolute value of the electron electric charge, s and s' are the spin states of the incoming and scattered electrons, respectively, and $Q^2 = -q^2$ is the four-momentum square of the virtual photon: $q = k - k'$. The formula above considers the QED interaction to be dominant which is true for a Q^2 value much smaller than the mass of the Z boson. Also, the quark current at spacetime point z has been defined as

$$j_\mu(z) = \sum_f \frac{e_f}{e} \bar{\mathrm{q}}_f(z) \gamma_\mu \mathrm{q}_f(z) \,, \tag{1.18}$$

making the flavor index f explicit and considering the color state implicit as in Eq. (1.8). Here, e_f is the electric charge of the quark f.

Taking into account that the set of states X is complete, we can apply

$$\sum_X \int \frac{d^3\vec{p}_X}{(2\pi)^3 2E_X} |X\rangle\langle X| = 1 \tag{1.19}$$

in the squared modulus of the amplitude (1.17) delivering the cross-section for unpolarized DIS:

$$\frac{d^2\sigma_{\text{DIS}}}{dx_B dy} = \frac{4\pi\alpha_{\text{EM}}^2 y}{Q^4} L^{\mu\nu} W_{\mu\nu} \,. \tag{1.20}$$

Note that the mass of the electron has been ignored. Here, $\alpha_{\text{EM}} = e^2/(4\pi)$ and we have introduced the *inelasticity variable*

$$y = \frac{pq}{pk} \underbrace{=}_{\substack{\text{proton}\\ \text{rest}\\ \text{frame}}} \frac{E_e - E'_e}{E_e} \,, \tag{1.21}$$

with E_e and E'_e the energy of the incoming and scattered electron beam in the proton rest frame, respectively. The variable x_B is referred to as the *Björken variable*:

$$x_B = \frac{Q^2}{2pq} \underbrace{=}_{\substack{\text{proton}\\ \text{rest}\\ \text{frame}}} \frac{Q^2}{2M(E_e - E'_e)} \,. \tag{1.22}$$

Here, M is the mass of the proton.

The symmetric lepton tensor $L^{\mu\nu}$ carries the information on the (massless) electron currents and can be expressed as

$$L^{\mu\nu} = k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(kk') \,, \tag{1.23}$$

with $g^{\mu\nu}$ the Minkowsky metric, while the hadron tensor $W_{\mu\nu}$ contains the details on the nucleon:

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4z \, e^{iqz} \langle p | [j_\mu(z), j_\nu(0)] | p \rangle \,. \tag{1.24}$$

Because the electric charge is preserved[1] by QED

$$q_\mu L^{\mu\nu} = q_\nu L^{\mu\nu} = 0 \quad \text{and} \quad q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0 \,. \tag{1.25}$$

These results together with parity conservation allows for a decomposition of the hadron tensor as

$$W_{\mu\nu} = \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}\right) W_1(x_B, Q^2) + \left(p_\mu + \frac{1}{2x_B} q_\mu\right)\left(p_\nu + \frac{1}{2x_B} q_\nu\right) W_2(x_B, Q^2) \,. \tag{1.26}$$

From here one reads (M the proton mass),

$$L^{\mu\nu} W_{\mu\nu} = -Q^2 \left[W_1(x_B, Q^2) - \frac{W_2(x_B, Q^2)}{2} \left(\frac{Q^2}{x_B^2} \frac{1-y}{y^2} - M^2 \right) \right] , \tag{1.27}$$

or alternatively

$$\frac{d^2\sigma_{\text{DIS}}}{dx_B dy} = \frac{4\pi\alpha_{\text{EM}}^2 y}{Q^2} \left[F_1(x_B, Q^2) + \frac{x_B F_2(x_B, Q^2)}{Q^2} \left(\frac{Q^2(1-y)}{x_B^2 y^2} - M^2 \right) \right] , \tag{1.28}$$

where we related the hadron components W_1 and W_2 to the *structure functions* (SFs) F_1 and F_2 via

$$F_1(x_B, Q^2) = -W_1(x_B, Q^2) \quad \text{and} \quad F_2(x_B, Q^2) = (pq) W_2(x_B, Q^2) \,. \tag{1.29}$$

As introduced earlier, QCD considers the existence of some fundamental degrees of freedom called partons that constitute the building blocks of hadrons such as the proton. Therefore, the photon-proton interaction should be interpreted by a photon-quark scattering. For a pointlike particle the SFs get reduced to the *elastic form factors* (EFFs) which are independent of the momentum transferred to the particle, in this case the photon momentum q. The connection between q-independent EFFs and pointlike particles comes from the Fourier transform of the EFFs which describes the electric and magnetic space-distributions of quarks, ρ_j^{quark}:

$$\rho_j^{\text{quark}}(\vec{r}, x_B) \sim \int d^3\vec{q}\, e^{i\vec{q}\vec{r}} F_j^{\text{quark}}(x_B) \sim \delta(\vec{r}) F_j^{\text{quark}}(x_B) \,. \tag{1.30}$$

Consequently, if the proton SFs are independent of q, i.e. $F_j(x_B, Q^2) \sim F_j(x_B)$, then the photon-proton scattering is the result of the photon striking pointlike particles. As the proton itself is an extended object [18], this would signal the existence of quarks. The first experiments conducting DIS and proving that for large virtuality ($Q^2 > M^2$) the FFs are independent of such momentum were performed by M. Breidenbach et al. at SLAC in 1969 [17]. This phenomenon is called *Björken scaling* and requires $Q^2, pq \to \infty$ while x_B finite, conditions known as the *Björken limit*.

As the proton is made out of partons, the SFs F_1 and F_2 of the proton ought to be related to the quark distributions. To find this relation we will connect the hadron tensor to matrix elements of quark operators.

An explicit calculation of the scattering matrix in perturbative field theory allows to

[1]Charge conservation as in Noether theorem $\partial_\mu j^\mu(z) = 0$ for the quark current in Eq. (1.18) would not apply in general in quantum field theories and it would be substitued by the Ward-Takahashi identities. However, the Standard Model, which QCD and QED are part of, is anomaly-free by design, meaning that Ward-Takahashi identities are formally equivalent to the Noether theorem.

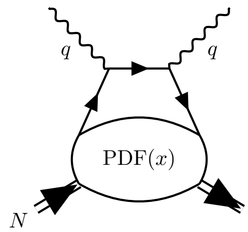

Fig. 1.3. Handbag diagram representing the photon-quark interaction in DIS that corresponds to the partonic interpretation of right graph in Fig. 1.2. Final and initial states coincide as a consequence of the optical theorem.

relate $W_{\mu\nu}$ to the amplitude of photon-quark interaction, or in other words to the so-called *handbag diagram*, vid. Fig. 1.3. Let us show this. To LO in the strong coupling constant, the expansion of the scattering matrix gives us:

$$e^2 4\pi VT\varepsilon^\mu(q,\Lambda)\,(\varepsilon^\nu(q,\Lambda))^*\,W_{\nu\mu} = \sum_X \int_X |\mathcal{T}_{\gamma N\to X}|^2 = 2\,\mathrm{Im}\{\mathcal{T}_{\gamma N\to\gamma N}\}\,. \tag{1.31}$$

Here, $\varepsilon^\mu(q,\Lambda)$ is the polarization vector of the photon with four-momentum q and polarization Λ, $VT = (2\pi)^4\delta(0)$ is the spacetime volume, $\sum_X$ stands for the sum over all possible particles in the set X and $\int_X$ is a shorthand for the momentum integration in Eq. (1.19). In the last equality, the optical theorem (1.15) was applied.

After some algebra, the amplitude of the handbag diagram, ignoring quark masses and to LO in α_s, reads

$$\begin{aligned} i\mathcal{T}_{\gamma N\to\gamma N} = &-e^2 VT\,\left(\varepsilon_\mu(q,\Lambda)\right)^*\varepsilon_\nu(q,\Lambda) \\ &\times \int d^4z\, e^{iqz}\langle p|\mathbb{T}\left\{j^\mu(z/2)j^\nu(-z/2)\right\}|p\rangle \\ &= VT\varepsilon^\mu(q,\Lambda)\,(\varepsilon^\nu(q,\Lambda))^*\sum_f e_f^2\int d^4\ell\left[\gamma_\mu\frac{i(\not\ell-\not q)}{(\ell-q)^2+i0}\gamma_\nu\right. \\ &\left.+\gamma_\nu\frac{i(\not\ell+\not q)}{(\ell+q)^2+i0}\gamma_\mu\right]_{ab}\times\int\frac{d^4z}{(2\pi)^4}\,e^{i\ell z}\langle p|{:}\mathrm{q}_f^b(z)\bar{\mathrm{q}}_f^a(0){:}|p\rangle\,, \end{aligned} \tag{1.32}$$

where ℓ is the momentum of the active parton and there is an implicit sum over the spinor indices a and b. Notice the use of the *normal ordering* of quark fields represented by two semicolons, to wit $:\mathrm{q}^b\bar{\mathrm{q}}^a:$. This is a consequence of the Wick theorem to solve the *time-ordered product* of the quark currents $\mathbb{T}\left\{j_\mu j_\nu\right\}$. From the amplitude above, the structure of the handbag diagram as illustrated in Fig. 1.3 is manifest: there are two polarization vectors for the incoming and outgoing photons, a propagator for the quark carrying momentum $\ell \pm q$ in-between the photon vertices (γ_μ, γ_ν insertions), and a loop integral over ℓ as well as a trace over the spinor indices a, b due to the close loop with the matrix element $\langle p|{:}\mathrm{q}_f^b(z)\bar{\mathrm{q}}_f^a(0){:}|p\rangle$. This one encapsulates the information on the parton content of the hadron and cannot be computed in perturbation theory.

The quark propagators in Eq. (1.32) can be split into their Cauchy's principal value (PV) and a Dirac delta

$$\frac{1}{x+i0} = \mathrm{PV}\left(\frac{1}{x}\right) - i\pi\delta(x)\,. \tag{1.33}$$

With the assumption that the photon virtuality is the scale of the process and, therefore, much larger than any other momentum, as well as realizing that the integral

$$\int \frac{d^4z}{(2\pi)^4}\, e^{i\ell z}\langle p|{:}\mathfrak{q}_f^b(z)\bar{\mathfrak{q}}_f^a(0){:}|p\rangle \tag{1.34}$$

is very damped for large values of ℓ, then we shall conclude that the principal values in Eq. (1.32) vanish. That the above integral is damped for large ℓ is a consequence of the Riemann-Lebesgue theorem which reads:

Theorem 1 (Riemann-Lebesgue) *Given a function $f(z) \in L^1(\mathbb{R}^n)$ and its Fourier transform*

$$\mathscr{F}(q) = \int_{\mathbb{R}^n} d^n z\, e^{iqz} f(z)\,; \tag{1.35}$$

if $|q^\mu| \to \infty$ for any μ, then

$$\mathscr{F} \to 0\,. \tag{1.36}$$

Here, $L^1(\mathbb{R}^n)$ is the Lebesgue space of functions that are integrable in module over $\mathbb{R}^n$, this is

$$f(z) \in L^1(\mathbb{R}^n) \Leftrightarrow \int_{\mathbb{R}^n} d^n z\, |f(z)| < \infty\,. \tag{1.37}$$

Hence, combining Eqs. (1.31), (1.32) and (1.33), we can read out the hadron tensor:

$$W_{\nu\mu} = -\sum_f \left(\frac{e_f}{e}\right)^2 \int d^4\ell\ \mathrm{Re}\{\gamma_\mu(\not{\ell} - \not{q})\delta((\ell-q)^2)\gamma_\nu + \gamma_\nu(\not{\ell} + \not{q})\delta((\ell+q)^2)\gamma_\mu\}_{ab}$$
$$\times \int \frac{d^4z}{(2\pi)^4}\, e^{i\ell z}\langle p|{:}\mathfrak{q}_f^b(z)\bar{\mathfrak{q}}_f^a(0){:}|p\rangle\,, \tag{1.38}$$

where we changed the imaginary part by the real component since for any complex number C, it holds $\mathrm{Im}\{iC\} = -\mathrm{Re}\{C\}$.

To address the Dirac deltas, we need to introduce the key concept of *light-cone coordinates* which will lead us to the so-called *collinear factorization*. The light-cone coordinates is a choice for decomposing any four-vector v. These coordinates are defined by means of two lightlike vectors n and n' such that $nn' \neq 0$. Then, v can be expressed as

$$v^\mu = v^+ n'^\mu + v^- n^\mu + v_\perp^\mu\,, \tag{1.39}$$

where $v^\pm$ are the longitudinal coordinates, whereas the perpendicular/transverse component $v_\perp$ is orthogonal to those directions: $v_\perp n = v_\perp n' = 0$. In this basis, the metric can be read out from the product of two four-vector v and w:

$$vw = (nn')(v^+w^- + v^-w^+) + v_{\perp,\mu}w_\perp^\mu \Rightarrow g_{+-} = g_{-+} = nn'\,, \tag{1.40}$$

where g_{+-}, g_{-+} represent the longitudinal components of the metric. For the transverse coordinates, the corresponding "perpendicular" metric is given by

$$g_{\perp,\mu\nu} = g_{\mu\nu} - \frac{n_\mu n'_\nu + n_\nu n'_\mu}{nn'}\,. \tag{1.41}$$

For example, if $n'^\mu = (1,0,0,1)/\sqrt{2}$ and $n^\mu = (1,0,0,-1)/\sqrt{2}$, then $g_{\perp,\mu\nu} = -1$ for $(\mu,\nu) \in \{(1,1),(2,2)\}$, while $g_{+-} = g_{-+} = 1$. Other matrix elements are zero.

It is usual to study parton interactions in the *infinite-momentum frame* where the mass of the proton is negligible compared to its three-momentum. In such case, we can choose the proton four-momentum as the vector defining the positive-longitudinal direction:

$$p^\mu = n'^\mu \,. \tag{1.42}$$

After selecting $pn = n'n = 1$ and $q_\perp^2 = q^2 = -Q^2$, the four-momentum of the photon is given by

$$q^\mu = \frac{Q^2}{2x_B} n^\mu + q_\perp^\mu \,. \tag{1.43}$$

Likewise, for the parton momentum:

$$\ell^\mu = x p^\mu + \frac{\ell^2 - \ell_\perp^2}{2x} n^\mu + \ell_\perp^\mu \,, \quad x \in (0,1)\,, \tag{1.44}$$

with x the parton fractional contribution to the total longitudinal momentum of the proton.

Hence, for the momentum squared in the Dirac deltas of Eq. (1.38) we have

$$\begin{aligned}(\ell + q)^2 &= \ell^2 - Q^2 + 2x\frac{Q^2}{2x_B} + 2\ell_\perp q_\perp \\ &= -Q^2\left(1 - \frac{x}{x_B}\right) + O\left(\frac{\ell^2}{Q^2}, \frac{\ell_\perp q_\perp}{Q^2}\right)\,,\end{aligned} \tag{1.45}$$

and

$$(\ell - q)^2 = -Q^2\left(1 + \frac{x}{x_B}\right) + O\left(\frac{\ell^2}{Q^2}, \frac{\ell_\perp q_\perp}{Q^2}\right)\,. \tag{1.46}$$

If we consider the hadron tensor (1.38) to be dominated by small ℓ^2 and small transverse dynamics ($\ell_\perp$, $q_\perp$) compared to the virtuality Q^2, then the process is dictated by the longitudinal kinematics which depend on x. These hypotheses are supported by the Riemann-Lebesgue theorem introduced earlier. In particular, the consideration of negligible transverse momenta is connected to the *light-cone dominance* of the hadron tensor, this is that the integral that defines the tensor (1.24) is dominated for light-cone distances: $z^2 \simeq 0$. Let us show that, indeed, this is a consequence of the Riemann-Lebesgue theorem.

From the definition of $W_{\mu\nu}$ in Eq. (1.24), the complex phase of the Fourier transform can be written according to the above momenta parameterization as

$$qz = \frac{Q^2}{2x_B} z^+ - \underbrace{\vec{q}_\perp \vec{z}_\perp}_{\substack{\text{Euclidean}\\ \text{product}}}\,, \tag{1.47}$$

where the product on the transverse vectors is in Euclidean space. For qz to be small, it requires $z^+ \sim 1/Q^2 \to 0$ (in Björken limit). Hence,

$$z^2 = 2z^+z^- - \vec{z}_\perp^{\,2} \xrightarrow[\text{limit}]{\text{Björken}} -\vec{z}_\perp^{\,2} \leq 0\,. \tag{1.48}$$

Since causality imposes $z^2 \geq 0$, the conclusion is straightforward: $z_\perp^\mu \simeq 0$ in Björken limit and so the hadron tensor is light-cone dominated, $z^2 \simeq 0$. The only remnant component is z^-, therefore the longitudinal components of four-vectors dominate the dynamics of processes such as DIS, *quod erat demonstrandum*.

Introducing Eqs. (1.45) and (1.46) in the hadron tensor of Eq. (1.38), as x is positive only one Dirac delta is left leading to the conclusion

$$x = x_B\,, \tag{1.49}$$

this is that the fraction of the proton longitudinal momentum that is carried by the parton, x, is equal to the Björken scale, x_B (at LO). Indeed, Eq. (1.38) simplifies to:

$$\begin{aligned} W_{\nu\mu} = &-\sum_f \left(\frac{e_f}{e}\right)^2 \int d^4\ell\ \mathrm{Re}\{\gamma_\nu(\not\ell + \not q)\gamma_\mu\}_{ab} \frac{\delta(x - x_B)}{2(pq)} \\ &\times \int \frac{d^4z}{(2\pi)^4}\, e^{i\ell z}\, \langle p|{:}\mathfrak{q}_f^b(z)\bar{\mathfrak{q}}_f^a(0){:}|p\rangle\,, \quad x = n\ell\,. \end{aligned} \tag{1.50}$$

With Eq. (1.29) we can obtain the FFs by projecting the hadron tensor with the vectors n and p. This way,

$$F_L = F_2 - 2x_BF_1 = \frac{(2x_B)^2}{pq} p^\nu p^\mu W_{\nu\mu}\,, \tag{1.51}$$

$$F_2 = (pq)n^\nu n^\mu W_{\nu\mu}\,, \tag{1.52}$$

where $F_L(x_B, Q^2)$ is called the *longitudinal form factor*. After some algebra, F_2 takes the form ($x = n\ell$)

$$\begin{aligned} F_2 &= -x_B \sum_f \left(\frac{e_f}{e}\right)^2 \int \frac{d^4\ell d^4z}{(2\pi)^4}\, e^{i\ell z}\ \mathrm{Re}\{\mathrm{tr}\,(\not n\langle p|{:}\mathfrak{q}_f(z)\bar{\mathfrak{q}}_f(0){:}|p\rangle)\}\delta(n\ell - x_B) \\ &= x_B \sum_f \left(\frac{e_f}{e}\right)^2 q_f(x_B)\,, \end{aligned} \tag{1.53}$$

where we have identified the *parton distribution function* (PDF) as

$$\begin{aligned} q_f(x_B) &= -\int \frac{d^4\ell d^4z}{(2\pi)^4}\, e^{i\ell z}\ \mathrm{Re}\{\mathrm{tr}\,(\not n\langle p|{:}\mathfrak{q}_f(z)\bar{\mathfrak{q}}_f(0){:}|p\rangle)\}\delta(n\ell - x_B) \\ &= \int \frac{d\lambda}{2\pi}\, e^{i\lambda x_B}\langle p|\bar{\mathfrak{q}}_f(0)\not n\mathfrak{q}_f(\lambda n)|p\rangle\,, \quad \lambda \in \mathbb{R}\,. \end{aligned} \tag{1.54}$$

Notice that, in the last expression, we removed the normal ordering of the quark-field product as it is not needed. In fact, the difference between having the normal ordering of the product and the product itself in the PDF definition is proportional to $\delta(rn + x_B)$, with r the four-momentum of the quark. The product rn is the plus-component in light-cone coordinates and, therefore, it is the fraction of the proton longitudinal momentum that the parton is carrying which is positive. As a result, $\delta(rn + x_B) = 0$ for all r.

Returning to the discussion on the FFs, we find that F_L is directly proportional to inverse powers of Q^2, so in the Björken limit it vanishes:

$$F_L \xrightarrow[\text{limit}]{\text{Björken}} 0 \Rightarrow F_2 = 2x_BF_1\,, \tag{1.55}$$

which is the well-known *Callan-Gross relation* and informs us that quarks are spin-1/2 particles.

The fact that the PDF q_f is a function of only x_B explains the Björken scaling observed in the experiments. The deviation with respect to the Björken scaling when the photon virtuality is not large enough as to ignore color interactions among quarks can be explained by the *Dokshitzer–Gribov–Lipatov–Altarelli–Parisi equation* (DGLAP) [19–21], which accounts for the effects on the quark distribution due to the emission and absorption of gluons.

When the PDF is introduced in the hadron tensor (1.50), the DIS cross-section (1.20) can be written as a convolution of the cross-section for the scattering electron-quark with the PDF:

$$\frac{d\sigma_{\text{DIS}}}{dy} = \sum_f \int_0^1 dx_B \, \frac{d^2\sigma_{e^- \mathrm{q}_f \to e^- X}}{dx_B dy} q_f(x_B)\,. \tag{1.56}$$

This factorization of the cross-section between a PDF (non-perturbative function) and the partonic cross-section (calculable in perturbation theory) is a feature of inclusive processes such as DIS. All the information on the internal structure of the hadron is contained in the PDF which depends on a single variable, x_B. Because of this, the PDF is a simplified picture of the hadron which is a byproduct of the missing information after the integration over the final states X, vid. Eqs. (1.19) and (1.31).

The lesson from DIS is that although color confinement prevents the isolation of partons, their distribution within the hadron is still accessible via high-energy experiments delivering a probe with a large momentum, ideally infinity. This probe is typically a virtual photon which interacts with partons. That this interaction is possible can be understood in the following way: the energy of the probe (given by the change in the energy of the electron beam during the scattering) is much larger than the typical energy exchange between partons via gluons, so that the time scale for the scattering probe-parton (inverse of the probe energy) is shorter than the parton-parton interaction. Hence, the constituents of the hadron appear as free states that can individually interact with the probe.

As indicated before, since the transverse dynamics do not play any role, the convolution (1.56) between a perturbative and a non-perturbative function is called collinear factorization. This is opposed to other processes where the transverse momentum is relevant and factorization happens via the *transverse-momentum parton distribution functions* (TMDs) which depend on both longitudinal and transverse components. Among those there are Drell-Yan and semi-inclusive DIS. For a review in TMDs and these processes cf. [22].

In order to go beyond the limited information of the PDFs within the picture of collinear factorization, one needs to consider *exclusive processes* for which we retain information on all incoming and outgoing particles. The first of this kind is called *deeply virtual Compton scattering* (DVCS) and was introduced by D. Müller, D. Robaschik, B. Geyer, F.-M. Dittes, J. Hořejši [23], X.-D. Ji [24] and A. V. Radyushkin [25] in the 1990s. In this process, an electron scatters off a nucleon which does not break apart but emits a real photon, see Fig. 1.4. The reaction in this case is

$$e^-(k) + N(p) \to e^-(k') + N(p') + \gamma(q')\,, \quad q'^2 = 0\,. \tag{1.57}$$

The photon-hadron scattering in DVCS consists of an incoming virtual photon with momentum $q = k - k'$ and a real one such that $q'^2 = 0$:

$$N(p) + \gamma^*(q) \to N(p') + \gamma(q')\,. \tag{1.58}$$

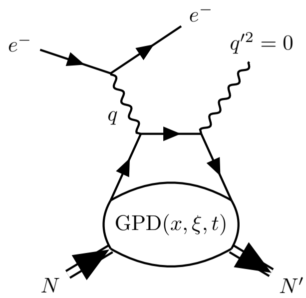

Fig. 1.4. Feynman diagram for deeply virtual Compton scattering (DVCS). Symbols N and N' are shorthands for $N(p)$ and $N(p')$, this is the same hadron but with different momentum.

Explicit calculation of the scattering matrix to LO in α_s renders:

$$i\mathcal{T}_{\gamma N\to\gamma N'} = ie^2(2\pi)^4\delta(p+k-p'-k'-q')\left(\varepsilon_\mu(q',\Lambda')\right)^*\varepsilon_\nu(q,\Lambda)T^{\mu\nu}_{s_2s_1}\,, \tag{1.59}$$

where Λ' and Λ stand for the polarizations of the virtual and real photons, respectively. Also, s_1 and s_2 are the spin quantum numbers for the initial- and final-state hadron for which $T^{\mu\nu}_{s_2s_1}$ represents the *Compton tensor*:

$$T^{\mu\nu}_{s_2s_1} = i\int d^4z\, e^{i\bar{q}z}\langle p',s_2|\mathbb{T}\left\{j_\mu(z/2)j_\nu(-z/2)\right\}|p,s_1\rangle\,, \quad \bar{q} = \frac{q+q'}{2}\,, \tag{1.60}$$

with the quark currents given in Eq. (1.18). After the use of the Wick theorem to deal with the time-ordered product of the currents, $\mathbb{T}\{j_\mu j_\nu\}$, the amplitude takes the form

$$\begin{aligned} i\mathcal{T}_{\gamma N\to\gamma N'} &= (2\pi)^4\delta(p+k-p'-k'-q')\left(\varepsilon^\mu(q',\Lambda')\right)^*\varepsilon^\nu(q,\Lambda) \\ &\quad\times\sum_f e_f^2\int d^4\ell\;\mathrm{tr}\Bigg\{\left[\gamma_\nu\frac{i(\not\ell-\not q')}{(\ell-q')^2+i0}\gamma_\mu+\gamma_\mu\frac{i(\not\ell+\not q)}{(\ell+q)^2+i0}\gamma_\nu\right] \\ &\quad\times\int\frac{d^4z}{(2\pi)^4}\,e^{i\ell z}\langle p',s_2|{:}\mathfrak{q}_f(z)\bar{\mathfrak{q}}_f(0){:}|p,s_1\rangle\Bigg\}\,. \end{aligned} \tag{1.61}$$

The Fierz completeness relation reads

$$\delta_{ac}\delta_{bd} = \frac{1}{4}\sum_i(\Gamma_i)_{ad}(\Gamma_i)_{bc}\,, \tag{1.62}$$

where δ_{ac}, δ_{bd} are Kronecker deltas, a, b, c, d are spinor indices and the set of Γ_i denotes the following Dirac structures: $\Gamma_S = 1$, $\Gamma_P = \gamma^5$, $\Gamma_V^\mu = \gamma^\mu$, $\Gamma_A^\mu = i\gamma^5\gamma^\mu$, $\Gamma_T^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] = \sigma^{\mu\nu}$. Making use of Eq. (1.62), we may write:

$$\begin{aligned} \mathfrak{q}_f^b(z)\bar{\mathfrak{q}}_f^a(0) &= \mathfrak{q}_f^d(z)\bar{\mathfrak{q}}_f^c(0)\delta_{bd}\delta_{ca} \\ &= \frac{1}{4}(\gamma^\mu)_{ba}\bar{\mathfrak{q}}_f(0)\gamma_\mu\mathfrak{q}_f(z) + \frac{1}{4}(\gamma^\mu\gamma^5)_{ba}\bar{\mathfrak{q}}_f(0)\gamma^5\gamma_\mu\mathfrak{q}_f(z) \\ &\quad + (\text{structures that give zero in the trace of Eq. (1.61)})\,. \end{aligned} \tag{1.63}$$

Sum over repeated indices is assumed. Upon introduction of this last expression into Eq. (1.61), the quark-matrix element exits the trace, yielding a separation of the amplitude into a vector ($\sim \bar{\mathrm{q}}\gamma^\mu \mathrm{q}$) and an axial-vector ($\sim \bar{\mathrm{q}}\gamma^5\gamma^\mu \mathrm{q}$) components:

$$\begin{aligned}
i\mathcal{T}_{\gamma N\to\gamma N'} = {} & \frac{(2\pi)^4}{4}\delta(p+k-p'-k'-q')\left(\varepsilon_\beta(q',\Lambda')\right)^*\varepsilon_\alpha(q,\Lambda) \\
& \times \sum_f e_f^2 \int d^4\ell \frac{d^4z}{(2\pi)^4}\, e^{i\ell z}\Bigg(\mathrm{tr}\bigg\{\left[\gamma^\alpha \frac{i(\not\ell-\not q')}{(\ell-q')^2+i0}\gamma^\beta+\gamma^\beta\frac{i(\not\ell+\not q)}{(\ell+q)^2+i0}\gamma^\alpha\right]\gamma^\mu\bigg\} \\
& \times \langle p', s_2|{:}\bar{\mathrm{q}}_f(0)\gamma_\mu \mathrm{q}_f(z){:}|p,s_1\rangle \\
& + \mathrm{tr}\bigg\{\left[\gamma^\alpha \frac{i(\not\ell-\not q')}{(\ell-q')^2+i0}\gamma^\beta+\gamma^\beta\frac{i(\not\ell+\not q)}{(\ell+q)^2+i0}\gamma^\alpha\right]\gamma^\mu\gamma^5\bigg\} \\
& \times \langle p', s_2|{:}\bar{\mathrm{q}}_f(0)\gamma^5\gamma_\mu \mathrm{q}_f(z){:}|p,s_1\rangle\Bigg)\,. \qquad (1.64)
\end{aligned}$$

In order to go further, we need to parameterize the momenta involved. For this purpose, as for the DIS case, we use light-cone coordinates and notice that the difference between p and p' can be described by the Mandelstam variable $t = \Delta^2 = (p'-p)^2$ and a positive parameter called *skewness* which is given by

$$\xi = -\frac{\Delta n}{2\bar{p}n} = \frac{x_B}{2-x_B} + O(|t|/Q^2) \quad \text{where} \quad \bar{p} = \frac{p+p'}{2}\,. \qquad (1.65)$$

This expression for ξ will be justified later on.

Selecting $q_\perp = p_\perp = 0$ and $nn' = 1$, a valid parameterization of the momenta involved in DVCS is

$$q^\mu = -2\xi\bar{p}^+ n'^\mu + \frac{Q^2}{4\xi\bar{p}^+}n^\mu\,, \qquad q'^\mu = -\frac{\Delta_\perp^2}{2q'^-}n'^\mu + q'^- n^\mu - \Delta_\perp^\mu\,, \qquad (1.66)$$

$$p^\mu = (1+\xi)\bar{p}^+ n'^\mu + \frac{M^2}{2(1+\xi)\bar{p}^+}n^\mu\,, \qquad p'^\mu = p'^+ n'^\mu + \frac{M^2-\Delta_\perp^2}{2p'^+}n^\mu + \Delta_\perp^\mu\,, \qquad (1.67)$$

with q'^- and p'^+ denoting

$$q'^- = \frac{Q^2}{4\xi\bar{p}^+} + \frac{M^2}{2(1+\xi)\bar{p}^+} - \frac{M^2-\Delta_\perp^2}{2p'^+}\,, \qquad (1.68)$$

$$p'^+ = (1-\xi)\bar{p}^+ + \frac{\Delta_\perp^2}{2q'^-}\,. \qquad (1.69)$$

From here we read out

$$\Delta^+ = (p'-p)^+ = \left[-2\xi + O(\Delta_\perp^2/Q^2)\right]\bar{p}^+\,, \qquad (1.70)$$

which suggests a parameterization of the parton initial and final momenta (ℓ and ℓ', respectively) of the form

$$\ell^\mu = (x+\xi)\bar{p}^+ n'^\mu + \ell^- n^\mu + \ell_\perp^\mu\,, \qquad (1.71)$$

$$\begin{aligned}
\ell'^\mu &= \ell^\mu + q^\mu - q'^\mu \\
&= (x-\xi)\bar{p}^+ n'^\mu + (\ell^- + q^- - q'^-)n^\mu + \ell_\perp^\mu + \Delta_\perp^\mu\,. \qquad (1.72)
\end{aligned}$$

Upon the Björken limit for which $Q^2 \to \infty$ while x_B is finite (so is ξ according to Eq. (1.65)), as well as neglecting transverse dynamics, it holds

$$\begin{aligned}(\ell+q)^\mu &= (x-\xi)\bar{p}^+ n'^\mu + \frac{Q^2}{4\xi\bar{p}^+}n^\mu\,, \qquad & (\ell+q)^2 &= \frac{Q^2}{2\xi}(x-\xi)\,,\\ (\ell-q')^\mu &= (x-\xi)\bar{p}^+ n'^\mu - \frac{Q^2}{4\xi\bar{p}^+}n^\mu\,, \qquad & (\ell-q')^2 &= -\frac{Q^2}{2\xi}(x-\xi)\,. \end{aligned} \tag{1.73}$$

As for the case of DIS, DVCS is light-cone dominated. The arguments given to justify the light-cone dominance of the DIS hadron tensor (1.24) apply to the Compton tensor (1.60) as well due to their similar structure. As a consequence, in DVCS all transverse momenta can be neglected in favor of the longitudinal components.

Introducing this parameterization on the propagators of Eq. (1.64), we are left with the traces:

$$\tau^{\beta\alpha\mu} = \mathrm{tr}\left\{\gamma^\beta \not{n}\gamma^\alpha\gamma^\mu\right\} \quad \text{and} \quad \widetilde{\tau}^{\beta\alpha\mu} = \mathrm{tr}\left\{\gamma^\beta \not{n}\gamma^\alpha\gamma^\mu\gamma^5\right\}\,. \tag{1.74}$$

For the DVCS case, the first one, τ, is non-zero in three scenarios:

1. Indices β and μ are transverse[2], whereas $\alpha = -$.
2. Indices α and μ are transverse, while $\beta = -$.
3. Indices β and α are transverse, but $\mu = -$.

The first two possibilities are incompatible with the longitudinal dominance over the transverse components as they impose $\gamma_{\perp,\mu}$ in the quark correlator of Eq. (1.64), which translate to $\bar{p}_{\perp,\mu}$, $\Delta_{\perp,\mu}$ in momentum space. A similar reasoning leads us to $\mu = -$ for the second trace, $\widetilde{\tau}$, too. Hence, we are left with

$$\tau^{\beta\alpha-} = -4g_\perp^{\beta\alpha} \quad \text{and} \quad \widetilde{\tau}^{\beta\alpha-} = 4i\epsilon_\perp^{\alpha\beta}\,, \tag{1.75}$$

where $\perp$ is used to denote that both indices, α and β, must refer to transverse coordinates. The perpendicular metric $g_\perp^{\beta\alpha}$ was already introduced in Eq. (1.41), whereas the perpendicular Levi-Civita tensor is

$$\epsilon_\perp^{\alpha\beta} = \epsilon^{\alpha\beta}{}_{\rho\sigma}n^\rho n'^\sigma\,, \tag{1.76}$$

with $\epsilon^{0123} = +1$.

Combining Eqs. (1.64), (1.73) and (1.75), the amplitude for the Compton scattering takes the factorized form:

$$\begin{aligned} i\mathcal{T}_{\gamma N\to\gamma N'} =& -ie^2(2\pi)^4\delta(p+k-p'-k'-q')\left(\varepsilon_\beta(q',\Lambda')\right)^*\varepsilon_\alpha(q,\Lambda)\\ &\times\sum_f\left(\frac{e_f}{e}\right)^2\int_{-1}^1 dx\left(\frac{g_\perp^{\beta\alpha}}{2}\left\{\frac{1}{x-\xi+i0}+\frac{1}{x+\xi-i0}\right\}F_f(x,\xi,t)\right.\\ &\left.+i\frac{\epsilon_\perp^{\beta\alpha}}{2}\left\{\frac{1}{x-\xi+i0}-\frac{1}{x+\xi-i0}\right\}\widetilde{F}_f(x,\xi,t)\right)\,. \end{aligned} \tag{1.77}$$

[2] In the context of light-cone coordinates we can also consider the decomposition of Dirac-gamma matrices as: $\gamma^\mu = \gamma^+ n'^\mu + \gamma^- n^\mu + \gamma_\perp^\mu$. For $nn' = 1$, it holds $\gamma^\pm = \gamma_\mp$.

From this last formula, we can identify the *generalized parton distributions* (GPDs). By decomposing the correlator F_f in a basis of vector spinor bilinears,

$$\begin{aligned} F_f(x,\xi,t) &= \int \frac{d\lambda}{2\pi}\, e^{ix(\bar{p}z)}\, \langle p', s_2 | \bar{\mathrm{q}}_f(-z/2) \not{n}\, \mathrm{q}_f(z/2) | p, s_1 \rangle \big|_{z=\lambda n} \\ &= \frac{1}{\bar{p}n} \left[H_f(x,\xi,t) \bar{u}(p',s_2) \not{n} u(p,s_1) + E_f(x,\xi,t) \bar{u}(p',s_2) \frac{i\sigma^{\alpha\beta} n_\alpha \Delta_\beta}{2M} u(p,s_1) \right] , \end{aligned} \tag{1.78}$$

we can read out the vector GPDs H and E. The parameter λ is integrated over $\mathbb{R}$. Likewise, there are axial GPDs $\widetilde{H}$ and $\widetilde{E}$ that come from:

$$\begin{aligned} \widetilde{F}_f(x,\xi,t) &= \int \frac{d\lambda}{2\pi}\, e^{ix(\bar{p}z)}\, \langle p', s_2 | \bar{\mathrm{q}}_f(-z/2) \not{n} \gamma^5 \mathrm{q}_f(z/2) | p, s_1 \rangle \big|_{z=\lambda n} \\ &= \frac{1}{\bar{p}n} \left[\widetilde{H}_f(x,\xi,t) \bar{u}(p',s_2) \not{n} \gamma^5 u(p,s_1) + \widetilde{E}_f(x,\xi,t) \bar{u}(p',s_2) \frac{\gamma^5 (n\Delta)}{2M} u(p,s_1) \right] . \end{aligned} \tag{1.79}$$

Since the correlators F_f and $\widetilde{F}_f$ are Lorentz invariant and so are the spinor bilinears contracted with n, then one concludes that the GPDs H, E, $\widetilde{H}$, $\widetilde{E}$ are Lorentz scalars. The vectors involved are p, p' and n, hence GPDs can only depend on these momenta through Δn, $\bar{p}n$ and $t = (p'-p)^2$ scalar variables. On top of this, the restriction on z to be along the $n-$direction implies that any Lorentz transformation on z is a dilation: $z \to \alpha z$, with α a constant, which is the same as to consider $n \to \alpha n$. Therefore, GPDs must be invariant under such a re-scaling so that they can only depend on the quotient between Δn and $\bar{p}n$. By selecting said quotient as

$$\xi = -\frac{\Delta n}{2\bar{p}n} = \frac{p^+ - p'^+}{p^+ + p'^+} , \tag{1.80}$$

it is granted to lay in the range $\xi \in (0,1]$. This way, we justify the definition of the skewness given in Eq. (1.65).

Notice that in Eqs. (1.78) and (1.79) we have removed the normal ordering that appeared in the matrix elements of Eq. (1.64). The difference is a term proportional to $\langle p', s_2 | p, s_1 \rangle$ which is zero for DVCS as $p' \neq p$.

The decomposition of the correlators above into a total of four GPDs is a feature of spin-1/2 targets. For the case of a (pseudo-)scalar target, i.e. spin-0, there is only one GPD, H.

For the case of DVCS, and in general of exclusive processes, the factorization happens at the level of the amplitude instead of the cross-section, as shown for the inclusive DIS. Notice that unlike PDFs, x in GPDs does not coincide with the Björken variable, although it still represents the average fraction of the hadron longitudinal momentum carried away by the active parton, vid. Eqs. (1.71) and (1.72).

The GPDs are off-forward[3] matrix elements of quark[4] operators that generalize the PDF in the sense that

$$H_f(x, \xi = 0, t = 0) = q_f(x) \quad \text{for } x > 0\,, \tag{1.81}$$

recovering this way the quark PDF. For $x < 0$, $H_f(x, 0, 0)$ is interpreted as the PDF for antiquarks. Time reversal conservation leads to

$$H_f(x, \xi, t) = H_f(x, -\xi, t)\,, \tag{1.82}$$

and by complex conjugation

$$\left(H_f(x, -\xi, t)\right)^* = H_f(x, \xi, t)\,, \tag{1.83}$$

so that we conclude that GPDs are real valued functions. The importance of these distributions is two-fold:

1. They are connected to the energy-momentum tensor of QCD [24] and, therefore, the total angular momentum of the hadron (orbital plus spin components) can be described by GPDs. This allows us to address the results of the European Muon Collaboration [27] where it was found that only around a 12% of the spin of the proton arises from the spin of its constituent quarks. The relation between the total angular momentum provided by quarks with favor f, J_f, and GPDs is given by the *Ji sum rule*:

$$J_f = \frac{1}{2} \int_{-1}^{1} dx\, x \left[H_f(x, \xi, 0) + E_f(x, \xi, 0) \right] \,. \tag{1.84}$$

2. The different GPDs have different interpretations by their Fourier transforms [28]. In particular, GPD H at $\xi = 0$ is associated with the probability of quarks carrying a given fraction, x, of the hadron longitudinal-momentum as a function of the position $\vec{b}_\perp$ in a plane perpendicular to the directions dictated by vectors n and n'. The Fourier transform linking both distributions is:

$$f(x, \vec{b}_\perp) = \int \frac{d^2\vec{\Delta}_\perp}{(2\pi)^2}\, e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp} H_f(x, 0, t = -\vec{\Delta}_\perp^2)\,. \tag{1.85}$$

   This is usually referred to as *hadron tomography*.

Because of these properties, GPDs constitute a fundamental part of the physics programs of major next-generation experimental facilities such as the US electron-ion collider (EIC) [29], the Chinese electron-ion collider (EIcC) [30] and the large hadron-electron collider (LHeC) [31], as well as of current and future experiments at the Jefferson Lab (JLab) [32].

The functions of $x \pm \xi$ that convolute with the GPDs in Eq. (1.77) are called *hard-coefficient functions* as they represents the photon-parton interaction and can be calculated in perturbation theory. These convolutions are called *Compton form factors*

[3] Off-forward means that the in- and out-states of the hadron which defines the matrix elements, this is the correlators F_f and $\widetilde{F}_f$, are not the same. Conversely, a PDF would be a forward matrix element.

[4] There are GPDs for gluons too, see for example [26].

(CFFs). To LO in the strong coupling constant and Björken limit, CFFs are defined as

$$\begin{aligned}\mathcal{H}(\xi,t) &= -\sum_f \left(\frac{e_f}{e}\right)^2 \int_{-1}^{1} dx \, \left(\frac{1}{x-\xi+i0} + \frac{1}{x+\xi-i0}\right) H_f(x,\xi,t) \\ &= -\int_{-1}^{1} dx \, \frac{1}{x-\xi+i0} H^{(+)}(x,\xi,t)\,, \end{aligned} \tag{1.86}$$

where the C-even[5] part of the GPD has been defined as

$$H^{(+)}(x,\xi,t) = \sum_f \left(\frac{e_f}{e}\right)^2 \left(H_f(x,\xi,t) - H_f(-x,\xi,t)\right)\,. \tag{1.87}$$

Mutatis mutandis we can define CFFs $\mathcal{E}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{E}}$ for the GPDs $E, \widetilde{H}, \widetilde{E}$, respectively. Making use of Eq. (1.33), the CFFs can be separated into its real and imaginary parts:

$$\mathcal{H}(\xi,t) = -\mathrm{PV}\int_{-1}^{1} dx \, \frac{1}{x-\xi} H^{(+)}(x,\xi,t) + i\pi H^{(+)}(\xi,\xi,t)\,, \tag{1.88}$$

so that the imaginary part of the CFF is given by the C-even part of the GPD at $x = \xi$. In literature it is custom to refer to $H^{(+)}$ as the GPD, although technically that would be H. In this manuscript, as long as there is not ambiguity, we will follow that denomination too.

To overcome the restriction $x = \xi$ for accessing GPDs in DVCS and map them in their whole kinematic domain, one could perform a deconvolution of the real part of the CFF or include next-to-leading-order (NLO) corrections in the strong coupling constant that would modify the hard-coefficient functions. Both attempts have their own drawbacks. On the one hand, deconvolution is not a mathematically well-defined operation, so following this approach to access GPDs is inherently affected by uncertainties. On the other hand, the inclusion of NLO corrections in DVCS is insufficient as a nuance appears: the *shadow GPDs* (SGPDs). Briefly, SGPDs consist of an infinitely large family of functions that can be added to GPD models without modifying the CFFs or the PDF limit [40] and so they are transparent to the observables. In order to solve these issues, one should consider other reactions. Among the different alternatives to DVCS, there exists *timelike Compton scattering* (TCS) proposed by E. R. Berger, M. Diehl and B. Pire [41] in the early 2000s. It can be understood as a complementary reaction to DVCS where a real photon scatters off a hadron producing a lepton-antilepton pair. In electron channel the reaction is:

$$\gamma(q) + N(p) \to N(p') + e^-(k) + e^+(k')\,, \quad q^2 = 0\,. \tag{1.89}$$

This process is depicted in Fig. 1.5. In this case, the energy scale correspond to the virtuality of the outgoing photon $q'^2 = Q'^2 > 0$ with $q' = k + k'$. In a similar manner as for the DVCS case, the amplitude is factorized by means of hard-coefficient functions and GPDs where now the CFFs are given by

$$\mathcal{H}(\xi,t) = -\mathrm{PV}\int_{-1}^{1} dx \, \frac{1}{x+\xi} H^{(+)}(x,\xi,t) + i\pi H^{(+)}(-\xi,\xi,t)\,. \tag{1.90}$$

[5] C-even is short for invariant under charge conjugation. To access the complementary C-odd part of quark GPDs, one needs to study other processes like diphoton photo- or electroproduction [33–36]. To access the chiral-odd quark GPDs, processes containing mesons in the final state are necessary [37–39].

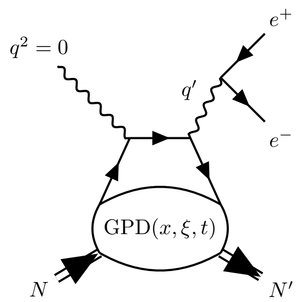

Fig. 1.5. Feynman diagram for timelike Compton scattering (TCS). Symbols N and N' are shorthands for $N(p)$ and $N(p')$, this is the same hadron but with different momentum.

This is equivalent to the CFF in DVCS (1.88) up to the reflection $\xi \to -\xi$. This modification does not provide new information in the GPDs as $H^{(+)}$ is odd in x. Nevertheless, TCS is useful to test the *universality* feature of GPDs, namely their independence of the process that is used to measure them. We remark that TCS has also a difference dependence on GPDs with respect to DVCS when NLO corrections are included, so data on both processes help constraining the information on GPDs better than either of them alone.

The restriction upon GPDs in DVCS and TCS to the lines $x = \pm\xi$ comes from the structure of the quark propagator in between the vertices of the real and the virtual photons. To modify such propagator we need to alter the photon states. The natural choice is to have two virtual photons instead of a real-virtual pair, as happens in DVCS and TCS. A process of this kind is *double deeply virtual Compton scattering* (DDVCS), firstly proposed by A. V. Belitsky, D. Müller [42], M. Guidal and M. Vanderhaeghen [43]. For DDVCS, an electron scatters off a hadron producing a lepton-antilepton pair. In muon channel, the reaction is:

$$e^-(k) + N(p) \to e^-(k') + N(p') + \mu^+(\ell_+) + \mu^-(\ell_-)\,, \tag{1.91}$$

and it is illustrated in Fig. 1.6. Notice that unlike DVCS or TCS, this process is a two-to-four scattering which enriches the kinematics. The incoming photon has momentum $q = k - k'$ with spacelike virtuality $Q^2 = -q^2 > 0$, whereas the emitted photon has momentum $q' = \ell_- + \ell_+$ and timelike virtuality $Q'^2 = q'^2 > 0$. The limit $Q^2 = 0$ renders TCS, while $Q'^2 = 0$ returns the DVCS results. As a consequence, DDVCS serves as a single framework to study simultaneously DVCS, TCS and DDVCS itself. The extra virtuality allows to define a new invariant named *generalized Björken variable*:

$$\rho = \xi \frac{qq'}{\Delta q'} = \xi \frac{Q^2 - Q'^2}{Q^2 + Q'^2} + O\left(\frac{\xi |t|}{Q^2 + Q'^2}\right)\,, \tag{1.92}$$

which can be both positive and negative depending on the relative magnitude between the two virtualities, but always satisfying $|\rho| \leq \xi$. In the Björken regime, DVCS is restored for $\rho \to \xi$, while TCS for $\rho \to -\xi$. By means of this new variable, the CFFs of DDVCS takes the form

$$\mathcal{H}(\rho, \xi, t) = -\text{PV} \int_{-1}^{1} dx\, \frac{1}{x - \rho} H^{(+)}(x, \xi, t) + i\pi H^{(+)}(\rho, \xi, t)\,, \tag{1.93}$$

providing access at LO to GPDs through the imaginary part of CFFs which turns out to be equivalent to the GPDs at $x = \rho$. As long as the photon virtualities are

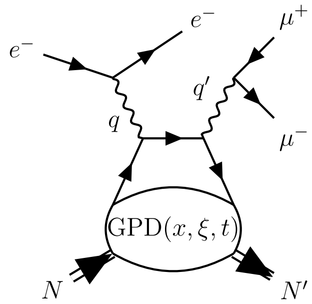

Fig. 1.6. Feynman diagram for double deeply virtual Compton scattering (DDVCS). Symbols N and N' are shorthands for $N(p)$ and $N(p')$, i.e. the same hadron but with different momentum.

different $\rho \neq \pm\xi$.

All results thus far have been obtained at LO in α_s and imposing the condition that the scale $\mathbb{Q}^2$ of the corresponding process ($\mathbb{Q}^2 = Q^2$ for DIS and DVCS, Q'^2 for TCS and $Q^2 + Q'^2$ for DDVCS) is ideally infinity. In experiments, the practical realization of this limit is done by considering $|t|/\mathbb{Q}^2 \ll 1$ as well as $M^2/\mathbb{Q}^2 \ll 1$, where M is the mass of the target. The cuts for rejecting data in order to fulfill these limits are somehow arbitrary [44] which introduce a bias when reconstructing GPDs. Corrections that are proportional to these kinematic factors are called *kinematic-twist corrections*, so that the limit when they are neglected (results so far) is known as the kinematic twist-2 approximation or, more commonly, the *leading twist*.

The Björken limit is in close relation to the parton operators being evaluated at lightlike distances, see Eqs. (1.54), (1.78) and (1.79) for which z is proportional to the lightlike vector n. This, in turn, is a consequence of the hadron (DIS) and Compton tensors (DVCS, TCS, DDVCS) being light-cone dominated. Deviation with respect to the light-cone in operators relaxes the Björken limit in the Fourier transform of their matrix elements, this is in the observables, prompting the twist corrections. Their study requires the techniques of the operator-product expansion, firstly introduced by K. G. Wilson in 1969 [45]. For our works, we will make use of the modern techniques based on conformal field theory and developed by V. M. Braun, Y. Ji and A. N. Manashov during the last decade [46, 47]. These techniques will be detailed later on in Ch. 3.

Accounting for these corrections improves the precision regarding nucleon tomography (1.85), which require data on a sizable range of the Mandelstam variable t. On top of this, extending the range of the scale $\mathbb{Q}^2$ will be useful for the analysis of the "mechanical" properties of partonic systems [48].

The aim of this doctoral thesis is to develop a formalism to extend the accessible domain of the kinematic variables x and t to reduce the theoretical uncertainty of exclusive processes. For that purpose, we will focus on DDVCS as it extends the range of x outside the lines $x = \pm\xi$ already at LO and serves as a single framework which in the appropriate limits reverts to DVCS and TCS. This feature is of special interest as there exist experimental data on DVCS and TCS. Although DDVCS has not been measured yet, there are experimental proposals at JLab [32] and the future US electron-ion collider (EIC) [29] planning on its measurement, so the study of this process is a timely matter. Among our results we present several observables that addresses the feasibility of DDVCS at those facilities, as well as estimate the size of the kinematic corrections in the CFFs that participate of DDVCS, DVCS and TCS.

These last two are studied as limiting cases of the first one.

PARTONS and EpIC softwares

The analytical results obtained in this doctoral project have been implemented in the PARTONS platform [49] and the EpIC Monte Carlo event generator [50] to study the feasibility of DDVCS measurements and the impact of the twist corrections. These codes have been publicly released in the PARTONS [51] and EpIC [52] repositories. In what follows we will describe these two softwares.

PARTONS is a C++ open-source platform with the intent of becoming a hub for theoretical developments and experimental applications in precision QCD. Its architecture is modular, i.e. the different functionalities of the program are encapsulated in independent components making PARTONS easy to update with new features. For example, the GPDModule class defines all the functions required to compute the GPDs. This base class is then used through C++ inheritance by a collection of classes that define specific GPD models (such as GPDGK11 for the Goloskokov-Kroll model described in Refs. [53, 54]). It also ensures consistent inputs and outputs throughout the code, with the GPD modules using GPDKinematics objects (storing the input GPD kinematics) and GPDResult objects (storing the computed values of GPDs). The use of the modular programming paradigm prevents code duplication and reduces mistakes, in particular by making the code easier to navigate.

Nowadays, PARTONS includes several types of GPD models such as Goloskokov-Kroll (GK) [53, 54], Vanderhaeghen-Guidal-Guichon (VGG) [55–58] and Mezrag-Moutarde-Sabatié (MMS) [59]. It uses APFEL++ [60] to solve the QCD evolution equations for various parton distributions. It also provides the evaluation of observables, such as cross-sections or spin and charge asymmetries for different types of exclusive processes, e.g. deeply vector meson production (DVMP), deeply virtual and timelike Compton scattering (DVCS and TCS), and double deeply virtual Compton scattering (DDVCS). In particular, the implementation of DDVCS at LO including higher-twist effects is one of the achievements of this doctoral thesis.

On the other hand, there is the EpIC Monte Carlo event generator. It is also a C++ open-source code based on modular programming and the PARTONS framework, making it fully compatible with the latter. Its modular nature enables the creation of multiple modules of the same type that can differ by computational algorithms or the physical assumptions considered for the generation. Hence, it provides a way to compare and select between them.

The current code of EpIC constitutes an advanced and fully functional version of the MC generator used in the analysis included in the EIC yellow report [29]. The generation by EpIC is highly precise as it includes QED radiative corrections [61, 62] and employs the FOAM library [63] for an accurate calculation of the cross-section. On top of this, EpIC allows for simulations in both collider and fixed-target kinematic setups, making it an ideal tool for experiments at the future EIC and the nowadays JLab, to give some examples.

Outline of this doctoral dissertation

For the Reader's convenience, in what follows we outline the contents of the next chapters. In chapter 2, we explore the electroproduction of a muon pair to access

GPDs through DDVCS off a spin-1/2 target. We develop a full formulation of the Feynman amplitudes of said process, getting rid of spinors and Dirac-gamma matrices thanks to the use of Kleiss-Stirling techniques [64, 65]. We employ our formalism to compute several observables in order to establish the feasibility of measuring DDVCS at current and future experimental facilities. To do so, we implement the amplitude and cross-section of DDVCS in `PARTONS` and `EpIC` softwares. Also, we study the limitations and consequences of the LT approximation and how to recover an electromagnetic-$U(1)$ invariant Compton tensor within the LT. In chapter 3, the theoretical background of the (conformal) twist decomposition is provided. This chapter is mostly based on the modern techniques of V. M. Braun, Y. Ji, A. N. Manashov, D. Müller and B. M. Pirnay [46, 47, 66–68] for the twist expansion. These techniques are exploited in the chapter 4 for the study of DDVCS off a spin-0 target. For this purpose, we parameterize the Compton tensor by means of helicity-dependent amplitudes (which are linked to CFFs) and compute the higher-twist corrections associated to each amplitude. As DVCS and TCS are limiting cases of DDVCS, from the formalism of the latter we are able to also deliver the calculation of DVCS (previously released in [66]) and TCS (novel). Furthermore, we provide numerical estimates of the helicity amplitudes. In the last chapter, we summarize our findings and their relevance for the field of QCD. This thesis also includes eleven appendices containing technical details of the calculations.

Works on DDVCS at LO and LT have been published in Refs. [69–72], and the higher-twist calculations and numerical analysis have been presented in the major physics conference "XXXI International Workshop on Deep Inelastic Scattering and Related Subjects" (DIS2024) [73] and part of proceedings (in preparation).

2

DDVCS off a nucleon target

In this chapter, we study the electroproduction of a muon pair on a spin-1/2 target N,

$$e^- + N \to e^- + N' + \mu^+ + \mu^- \,, \tag{2.1}$$

at leading order (LO) in the strong coupling constant and leading twist (LT) accuracy. We use N' to represent the same hadron N after being struck by the electron beam. Our interest in this process lays on the possibility of accessing GPDs at $x \neq \pm\xi$ already at LO, as opposed to DVCS and TCS [41, 74].

First, we present a full parameterization of the reaction (2.1), following the reference frames of [42], and establish the relations to other frames more popular on GPD phenomenology: the Trento [75] and the Berger-Diehl-Pire frames [41]. In a next section, we detail the calculation of the Feynman amplitudes and the cross-section of reaction (2.1) by means of the Kleiss-Stirling techniques [64, 65], including polarization of the target N. This formulation is cross-checked by comparing the small incoming and outgoing virtuality limits of the process (2.1) with the TCS and DVCS cases, respectively. We code our formulas in the `PARTONS` [49, 51] and `EpIC` [50, 52] softwares, vid. Sect. 1, to render predictions on different observables for current (JLab) and future (EIC) experiments. Our results support the feasibility of measuring DDVCS in both facilities.

A byproduct of this analysis is the observation of the frame dependence of the leading-twist approximation, theoretically explained in [68]. We detail how to reduce the phenomenological impact of the frame choice and how to restore the electromagnetic gauge invariance (violated at LT) in a way consistent with collinear factorization.

2.1 Description of DDVCS

In this chapter we explore the exclusive electroproduction of a muon pair on a proton target,

$$e^-(k) + N(p) \to e^-(k') + N(p') + \mu^+(\ell_+) + \mu^-(\ell_-) \,, \tag{2.2}$$

which receives contributions from two subprocesses with the same initial and final states. One of them is the so-called *Bethe-Heitler subprocess* (BH, middle and right diagrams in Fig. 2.1), which is a pure QED scattering. In this process, the photon-proton

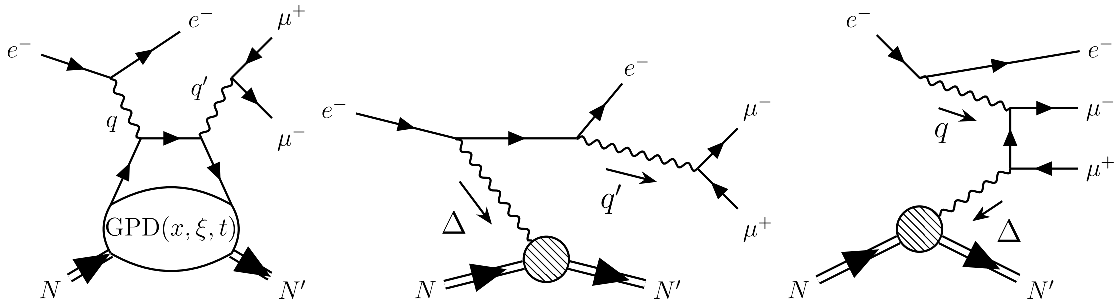

Fig. 2.1. From left to right: double deeply virtual Compton scattering (DDVCS) process at leading order and the two types of Bethe-Heitler (BH) processes, which contribute to the electroproduction of a muon pair. Complementary crossed diagrams are not shown in this figure. We use N' as a shorthand for $N(p')$.

interaction consists of a single non-elementary vertex that can be parameterized by elastic form factors (EFFs) only: it does not provide access to GPDs.

The other subprocess is double deeply virtual Compton scattering (DDVCS, left graph in Fig. 2.1) which was briefly presented in Ch. 1. This subprocess is a Compton scattering where the proton absorbs and emits (virtual) photons:

$$\gamma^*(q) + N(p) \to \gamma^*(q') + N(p')\,. \tag{2.3}$$

Here, $q = k - k'$ and $q' = \ell_- + \ell_+$. The squared momentum of the incoming photon is usually referred to as the *spacelike virtuality* $Q^2 = -q^2 = -(k-k')^2 > 0$, while that of the outgoing photon as the *timelike virtuality* $Q'^2 = q'^2 = (\ell_- + \ell_+)^2 > 0$. In the Björken limit, where the sum of both virtualities is ideally infinite while the skewness parameter ξ (1.65) remains finite, DDVCS factorizes into coefficient functions which can be calculated in perturbation theory representing the photon-parton interaction and non-perturbative terms, the GPDs which encompasses the information on the three-dimensional structure of the proton.

We will focus on the muon production on DDVCS, as indicated in Eq. (2.2), for two reasons:

1. To avoid extra Feynman diagrams accounting for the crossing of the electron coming from the beam and the produced electron (Pauli exclusion principle).

2. In future experimental measurements of DDVCS, distinguishing between the electron from the produced lepton pair and the one from the scattered beam is a difficult task that would introduce extra uncertainties with respect to a measurement where muons are detected.

Our study of DDVCS will be perform at leading-order (LO) in the strong coupling constant and in a kinematic region compatible with the Björken limit, that is for small ratios $|t|/(Q^2 + Q'^2)$ and $M^2/(Q^2 + Q'^2)$ where $t = \Delta^2 = (p' - p)^2 < 0$ is the Mandelstam variable and M is the mass of the proton. This means we stay at kinematic twist-2, more commonly known as the leading twist (LT). Upon these conditions, this chapter focuses on the first goal of this thesis, namely the access to $x \neq \pm\xi$, overcoming the restriction of DVCS and TCS at their lowest order. The second goal, this is the expansion on the accessible t-range is left for Chs. 3 and 4. The latter will require the inclusion of kinematic power corrections of the form $|t|/(Q^2 + Q'^2)$ and $M^2/(Q^2 + Q'^2)$, i.e. to include higher-twist contributions.

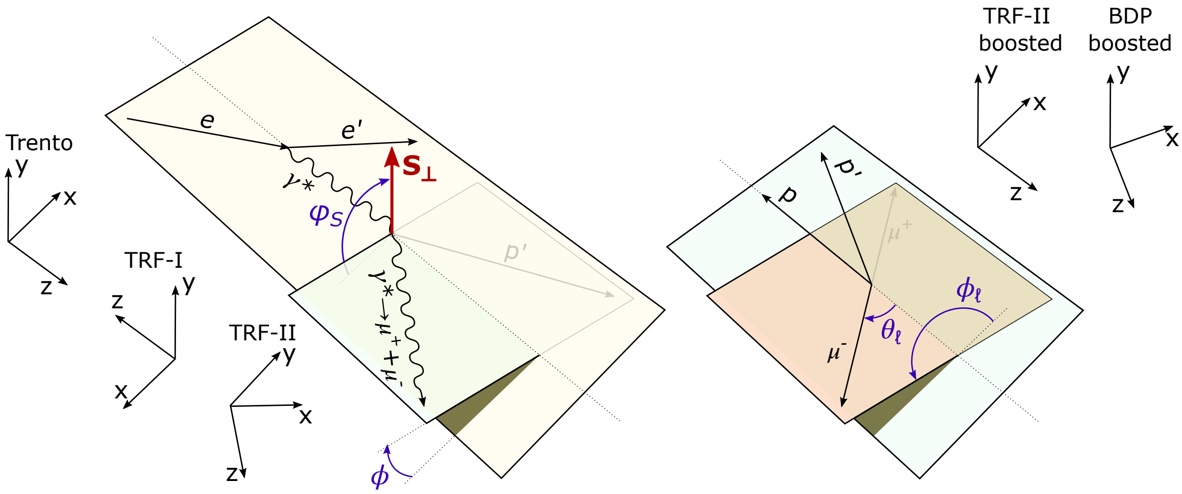

Fig. 2.2. Kinematics of DDVCS process with the depiction of coordinate systems discussed in the text. On the left: Frame with initial proton at rest and transverse component of target polarization vector, $\vec{S}_\perp$, defined with respect to the incoming virtual photon. On the right: Produced lepton pair center of mass frame.

This chapter is organized as follows: first, we introduce the relevant variables and invariants that we use to describe the DDVCS process. In a next section, we compute the amplitudes for both the BH subprocess (consisting of four Feynman diagrams) and the DDVCS (two diagrams) by means of the Kleiss-Stirling spinor techniques. In a next step, we implement the overall cross-section in the open-source `PARTONS` software and use it to compute the TCS and DVCS limits by considering $Q^2 \to 0$ and $Q'^2 \to 0$, respectively. This code is also used to discuss several DDVCS observables and tentatively estimate the feasibility of measuring this process in current (JLab) and future (EIC) experimental facilities. The `EpIC` Monte Carlo generator, which uses the `PARTONS` framework, is used to estimate the most favorable kinematic region for measuring DDVCS.

2.2 Reference frames and momenta parameterization

In this section we describe the kinematics of the electroproduction of a muon pair (2.2). The core of our evaluation is done in reference frames that coincide with those used in Ref. [76]. This choice allows us to stay consistent with the literature on the DDVCS topic, but also to facilitate comparison of obtained results. The kinematic variables, momenta and reference frames used throughout this work are depicted in Fig. 2.2.

We start the discussion by defining the *target rest frame I* (TRF-I), where the initial-state proton stays at rest, and where the $z-$axis is counter-aligned with respect to the incoming photon direction. A 180° clockwise rotation around the $y-$axis recovers the Trento frame [75], commonly used in GPD phenomenology. In TRF-I the four-momenta of the initial-state proton and of the incoming photon are:

$$\mathrm{p}^\mu = (M,0,0,0)\,, \quad \mathrm{q}^\mu = (\mathrm{q}^0,0,0,\mathrm{q}^3)\,, \quad \mathrm{q}^3 < 0\,. \tag{2.4}$$

Selecting the incoming photon virtuality $Q^2 = -\mathrm{q}^2$ and the Björken variable x_B (1.22) as parameters, one can fully determine q:

$$\mathrm{q}^\mu = \frac{Q}{\varepsilon}\left(1, 0, 0, -\sqrt{1+\varepsilon^2}\right), \tag{2.5}$$

where ε is:

$$\varepsilon = \frac{2x_B M}{Q}. \tag{2.6}$$

The scattered proton momentum can be written as

$$\mathrm{p}'^\mu = \left(M - \frac{t}{2M}, |\vec{\mathrm{p}}'|\sin\theta_N\cos\phi, |\vec{\mathrm{p}}'|\sin\theta_N\sin\phi, |\vec{\mathrm{p}}'|\cos\theta_N\right). \tag{2.7}$$

With the previous parameterization of p and using the Mandelstam variable $t = \Delta^2 = (\mathrm{p}' - \mathrm{p})^2$, we can identify p'^0 and $|\vec{\mathrm{p}}'|$ as

$$\mathrm{p}'^0 = M - \frac{t}{2M}, \quad |\vec{\mathrm{p}}'| = \sqrt{-t\left(1 - \frac{t}{4M^2}\right)}. \tag{2.8}$$

The angles ϕ and θ_N represent the azimuthal and polar directions for the scattered proton in TRF-I. In particular, ϕ is measured while θ_N can be determined by the angle between $\vec{\mathrm{p}}'$ and $\vec{\mathrm{q}}$ as the latter is opposite to the $z-$axis:

$$\cos\theta_N = -\frac{\varepsilon^2(Q^2 + Q'^2 - t) - 2x_B t}{4x_B M|\vec{\mathrm{p}}'|\sqrt{1+\varepsilon^2}}. \tag{2.9}$$

The momentum of the outgoing virtual photon is:

$$\mathrm{q}'^\mu = \mathrm{q}'^0(1, \vec{v}), \tag{2.10}$$

where q'^0 can be read out from t

$$t = (\mathrm{p}' - \mathrm{p})(\mathrm{q} - \mathrm{q}') = M\left[\frac{t}{2M} - \frac{Q}{\varepsilon} + \mathrm{q}'^0\right] \Rightarrow \mathrm{q}'^0 = \frac{Q}{\varepsilon} + \frac{t}{2M}, \tag{2.11}$$

and $|\vec{v}|$ from value of the timelike virtuality

$$Q'^2 = \mathrm{q}'^2 \Rightarrow |\vec{v}| = \sqrt{1 - \left(\frac{Q'}{\mathrm{q}'^0}\right)^2}. \tag{2.12}$$

The vector $\vec{v}$ can be interpreted as the "speed" of the outgoing virtual photon in the sense of App. A.

With hadron and photon momenta settled, we focus now on the leptons. The incoming electron beam moves in the $(x, z)-$plane of TRF-I, so for a massless electron we have:

$$\mathrm{k}^\mu = \mathrm{k}^0(1, \sin\theta_e, 0, \cos\theta_e), \tag{2.13}$$

where k^0 can be related to the inelasticity y (1.21) and the Björken variables x_B (1.22)

$$\mathrm{k}^0 = \frac{Q}{\varepsilon y}. \tag{2.14}$$

The polar angle of the beam, θ_e, is determined by the product between $\vec{\mathtt{k}}$ and $\vec{\mathtt{q}}$ as

$$\cos\theta_e = \frac{-1}{\sqrt{1+\varepsilon^2}}\left(1+\frac{y\varepsilon^2}{2}\right). \tag{2.15}$$

The momentum of the scattered electron is given by momentum conservation, $\mathtt{k}' = \mathtt{k} - \mathtt{q}$. The momenta of the muon-antimoun pair are easier to calculated in their center of mass frame and they can be related to the momentum of the outgoing photon $\mathtt{q}'$. Therefore, it seems more convenient to change the reference frame to one where the $z-$axis is aligned with the three-momentum of said photon. This frame is called *target mass frame II* (TRF-II) and its relation to TRF-I is given by a rotation of an angle θ_γ between their $z-$axes. Therefore, θ_γ can be parameterized by t provided that $\vec{q}\cdot\vec{q}\,' = -|\vec{q}|\cdot|\vec{q}\,'|\cos\theta_\gamma$:

$$\cos\theta_\gamma = -\frac{\varepsilon(Q^2 - Q'^2 + t)/2 + Q\mathtt{q}'^0}{Q|\vec{v}|\mathtt{q}'^0\sqrt{1+\varepsilon^2}}. \tag{2.16}$$

After some algebra, the Lorentz transformation between TRF-I and TRF-II is

$$\mathcal{R}_{\mathrm{II}\leftarrow\mathrm{I}} = \left(\begin{array}{c|c} 1 & 0_{1\times3} \\ \hline 0_{3\times1} & (R_{\mathrm{II}\leftarrow\mathrm{I}})_{3\times3}\end{array}\right), \quad (R_{\mathrm{II}\leftarrow\mathrm{I}})_{3\times3} = \begin{pmatrix} -c_\gamma c_\phi & -c_\gamma s_\phi & -s_\gamma \\ s_\phi & -c_\phi & 0 \\ -s_\gamma c_\phi & -s_\gamma s_\phi & c_\gamma \end{pmatrix}, \tag{2.17}$$

where the shorthands $c_\gamma = \cos\theta_\gamma$, $c_\phi = \cos\phi$, and likewise for sines, are used. Applying this transformation to the momenta displayed above for TRF-I, one has in TRF-II:

$$q'^\mu = \mathtt{q}'^0(1,0,0,|\vec{v}|)\,, \tag{2.18}$$
$$q^\mu = (\mathtt{q}^0, -s_\gamma\mathtt{q}^3, 0, c_\gamma\mathtt{q}^3)\,, \tag{2.19}$$
$$k^\mu = E_e(1, -s_e c_\gamma c_\phi - c_e s_\gamma, s_e s_\phi, -s_e s_\gamma c_\phi + c_e c_\gamma)\,, \tag{2.20}$$
$$p^\mu = \mathtt{p}^\mu\,, \tag{2.21}$$

where E_e is the energy of the incoming electron beam, $\mathtt{q}^0, \mathtt{q}^3$ and $\mathtt{q}'^0$ can be read out from Eqs. (2.5) and (2.11), and $s_e = \sin\theta_e$, $c_e = \cos\theta_e$ as given by Eq. (2.15).

The muon mass (m_ℓ) effects come in powers of m_ℓ^2/Q'^2. As Q'^2 is typically of the order of the GeV^2 (to avoid resonances), we can consider muons as massless particles henceforth. Consequently, in the center of mass frame of the muon pair we can write the momenta of the muon (ℓ_-) and the antimuon (ℓ_+) as lightlike vectors of the form:

$$\ell^\mu_{\mp,\mathrm{CM}} = \frac{Q'}{2}(1, \pm\vec{\beta})\,, \quad \vec{\beta} = (\sin\theta_\ell\cos\phi_\ell, \sin\theta_\ell\sin\phi_\ell, \cos\theta_\ell)\,, \tag{2.22}$$

where ϕ_ℓ, θ_ℓ are the azimuthal and polar directions of these particles with respect to their center of mass frame. Boosting back to TRF-II using $\vec{v}$ (2.12), we get

$$\ell^\mu_- = \left(\frac{1}{2}\mathtt{q}'^0(1+|\vec{v}|\cos\theta_\ell), \frac{1}{2}Q'\sin\theta_\ell\cos\phi_\ell, \frac{1}{2}Q'\sin\theta_\ell\sin\phi_\ell, \frac{1}{2}\mathtt{q}'^0(|\vec{v}|+\cos\theta_\ell)\right), \tag{2.23}$$

and ℓ_+ is obtained by the substitution $(\phi_\ell, \theta_\ell) \to (\pi+\phi_\ell, \pi-\theta_\ell)$.

Throughout the momenta parameterization, we chose x_B, t, y (or, equivalently, the electron energy E_e (1.21)), the scattered proton azimuthal angle (ϕ), the muon azimuthal and polar directions (ϕ_ℓ, θ_ℓ) and the photon virtualities (Q^2, Q'^2) as parameters. Therefore, these variables describe the phase-space to be measure in experiments in order to fulfill the *exclusivity* conditions of DDVCS, i.e. to have a complete knowledge of all in and out states as required by GPD factorization.

For later use, we also specify the range of the Mandelstam variable t allowed by the kinematics of the process:

$$t_0(t_1) = -\frac{1}{4x_B(1-x_B)+\varepsilon^2}\left\{2[(1-x_B)Q^2 - x_B Q'^2] + \varepsilon^2(Q^2 - Q'^2) \right. \\ \left. \mp 2\sqrt{1+\varepsilon^2}\sqrt{[(1-x_B)Q^2 - x_B Q'^2]^2 - \varepsilon^2 Q^2 Q'^2}\right\}, \tag{2.24}$$

where $t_0(t_1)$ corresponds to $-(+)$ sign and the minimal (maximal) absolute value of t: $|t_1| \geq |t| \geq |t_0|$. Note that $\Delta_\perp^2 \to 0$ as $t \to t_0$.

2.3 Relations to Trento and BDP frames

In literature, it is common to describe observables in the *Trento frame* [75], typical of the DVCS process, as well as in the *Berger-Diehl-Pire frame* (BDP) [41], popular in TCS analyses. As a consequence, we designed the DDVCS modules in `PARTONS` to receive inputs and produce outputs compatible with these two frames. In particular, we relate the azimuthal direction followed by the scattered proton ϕ and the azimuthal angle of the target polarization vector $\vec{S}$, which are described in TRF-I cf. Fig. 2.2, to its Trento value. In a similar manner, we establish the relation between the muon-pair angles ϕ_ℓ, θ_ℓ in boosted[1] TRF-II and their BDP equivalent.

Rotating 180º clockwise around the $y-$axis of TRF-I recovers the Trento frame, see plot in the right of Fig. 2.2. Hence, ϕ and φ_S are:

$$\phi = \begin{cases} \pi - \phi_{\text{Trento}}, & \text{if } \phi_{\text{Trento}} \in [0,\pi] \\ 3\pi - \phi_{\text{Trento}}, & \text{if } \phi_{\text{Trento}} \in (\pi, 2\pi) \end{cases}, \tag{2.25}$$

$$\varphi_S = \begin{cases} \pi - \varphi_{S,\text{Trento}}, & \text{if } \varphi_{S,\text{Trento}} \in [0,\pi] \\ 3\pi - \varphi_{S,\text{Trento}}, & \text{if } \varphi_{S,\text{Trento}} \in (\pi, 2\pi) \end{cases}. \tag{2.26}$$

Notice that the BDP frame is defined with its $z-$axis opposite to the three-momentum of the final-state proton and in the TCS limit it coincides with TRF-II. In DDVCS, since the incoming photon virtuality is non-zero, p' acquires a non-zero p'^1 coordinate so that the boosted BDP and TRF-II are related by a rotation around the $y-$axis, see plot in the left of Fig. 2.2, given by an angle that we denote as χ. To find χ one needs to boost p' to the center of mass of the muon pair using the rapidity

$$\zeta = \text{arctanh}\,|\vec{v}|, \tag{2.27}$$

[1]To the center of mass of the produced muon-antimuon system.

where $|\vec{v}|$ is given in Eq. (2.12), and perform the rotation by χ fixing the condition $p'^1 = 0$. The result is

$$c_\chi = \frac{p'^0 \sinh\zeta - p'^3 \cosh\zeta}{\sqrt{(p'^1)^2 + (p'^0 \sinh\zeta - p'^3 \cosh\zeta)^2}}, \tag{2.28}$$

$$s_\chi = \frac{p'^1}{\sqrt{(p'^1)^2 + (p'^0 \sinh\zeta - p'^3 \cosh\zeta)^2}}, \tag{2.29}$$

where c_χ and s_χ are shorthands for $\cos\chi$ and $\sin\chi$, respectively. The p' coordinates can be read out from Eqs. (2.18), (2.19) and (2.21), since momentum conservation imposes $p' = p + q - q'$.

With c_χ and s_χ above, we can perform the rotation of the muon momentum in Eq. (2.22). Taking into account that rotations do not alter the modulus of the three-momentum, we have that in the BDP frame:

$$\begin{aligned}\ell^\mu_{-,\underset{\text{BDP}}{\text{CM}}} &= \frac{Q'}{2}\,(1,\, \sin\theta_{\ell,\text{BDP}}\cos\phi_{\ell,\text{BDP}},\, \sin\theta_{\ell,\text{BDP}}\sin\phi_{\ell,\text{BDP}},\, \cos\theta_{\ell,\text{BDP}}) \\ &= \frac{Q'}{2}(1,\, c_\chi \sin\theta_\ell\cos\phi_\ell + s_\chi\cos\theta_\ell,\, \sin\theta_\ell\sin\phi_\ell,\, -s_\chi\sin\theta_\ell\cos\phi_\ell + c_\chi\cos\theta_\ell)\,.\end{aligned} \tag{2.30}$$

From here we obtain:

$$\sin\theta_{\ell,\text{BDP}} = \sqrt{(c_\chi\sin\theta_\ell\cos\phi_\ell + s_\chi\cos\theta_\ell)^2 + \sin^2\theta_\ell\sin^2\phi_\ell}\,, \tag{2.31}$$

$$\sin\phi_{\ell,\text{BDP}} = \sin\theta_\ell\sin\phi_\ell/\sin\theta_{\ell,\text{BDP}}\,, \tag{2.32}$$

$$\cos\phi_{\ell,\text{BDP}} = (c_\chi\sin\theta_\ell\cos\phi_\ell + s_\chi\cos\theta_\ell)/\sin\theta_{\ell,\text{BDP}}\,, \tag{2.33}$$

which relates TRF-II to BDP.

In what follows, we will work in the TRF-II frame but present our phenomenological predictions in BDP and Trento angles as they are the custom in experiments.

2.4 Amplitudes and cross-section

In Sect. 2.2, we described the kinematics involved in the reaction (2.2) and identify the phase-space as dependent on eight variables: E_e, x_B, t, ϕ, ϕ_ℓ, θ_ℓ, Q^2 and Q'^2. As the energy of the electron beam E_e is fixed we can use the remnant seven to describe the *Lorentz invariant phase-space* (LIPS) of the scattering cross-section σ. Provided the decomposition of the scattering matrix as

$$\mathcal{S} = 1 + i\mathcal{T}\,, \tag{2.34}$$

the interaction matrix $\mathcal{T}$ is related to the Feynman amplitude $i\mathcal{M}$ via

$$i\mathcal{T} = (2\pi)^4\delta\left(\sum_I p_I - \sum_F p_F\right) i\mathcal{M}\,, \tag{2.35}$$

where p_I and p_F describe the set of incoming and outgoing four-momenta. The cross-section for (2.2) can be written by means of the corresponding Feynman amplitude as:

$$d\sigma = \frac{|\mathcal{M}|^2}{4(pk)} d\text{LIPS}_4 \,, \tag{2.36}$$

where lepton masses are neglected and

$$\begin{aligned} d\text{LIPS}_4 = (2\pi)^4 \delta(p + q - p' - q') \frac{d^4 p'}{(2\pi)^3} \frac{d^4 k'}{(2\pi)^3} \frac{d^4 \ell_-}{(2\pi)^3} \frac{d^4 \ell_+}{(2\pi)^3} \\ \times \delta_+(p'^2 - M^2) \delta_+(k'^2) \delta_+(\ell_-^2) \delta_+(\ell_+^2) \,. \end{aligned} \tag{2.37}$$

Here, we considered the notation:

$$\delta_+(X^2 - Y^2) = \delta(X^2 - Y^2)\theta(X^0) \,, \tag{2.38}$$

with $\theta(\cdot)$ is the Heaviside step function.

After the change of variable $\ell_+ = q' - \ell_-$ and introducing the physical constraint $q'^2 = Q'^2$ by the identity

$$\int d(Q'^2) \delta_+(q'^2 - Q'^2) = 1 \,, \tag{2.39}$$

we are left with

$$d\text{LIPS}_4 = dQ'^2 \times d\text{LIPS}_3 \times d\text{LIPS}_{\mu^-\mu^+} \,, \tag{2.40}$$

where the LIPS for the outgoing electron-photon-proton system is

$$\begin{aligned} d\text{LIPS}_3 = (2\pi)^4 \delta(p + k - k' - q' - p') \\ \times \frac{d^4 p'}{(2\pi)^3} \delta_+(p'^2 - M^2) \frac{d^4 k'}{(2\pi)^3} \delta_+(k'^2) \frac{d^4 q'}{(2\pi)^3} \delta_+(q'^2 - Q'^2) \,, \end{aligned} \tag{2.41}$$

and that of the muon pair is

$$d\text{LIPS}_{\mu^-\mu^+} = \frac{d^4 \ell_-}{(2\pi)^3} \delta_+(\ell_-^2) \delta_+((q' - \ell_-)^2) \,. \tag{2.42}$$

Finally, we obtain:

$$d\text{LIPS}_3 = \frac{1}{32 x_B M E_e (2\pi)^4 \sqrt{1 + \varepsilon^2}} dx_B dQ^2 d|t| d\phi \,, \tag{2.43}$$

and

$$d\text{LIPS}_{\mu^-\mu^+} = \frac{1}{8(2\pi)^3} d\Omega_\ell \,, \quad d\Omega_\ell = \sin\theta_\ell d\phi_\ell d\theta_\ell \,. \tag{2.44}$$

Finally, the cross-section takes the form:

$$\frac{d^7\sigma}{dx_B dQ^2 dQ'^2 d|t| d\phi d\Omega_\ell} = \frac{\alpha_{\text{EM}}^4}{16(2\pi)^3} \frac{x_B y^2}{Q^4 \sqrt{1 + \varepsilon^2}} \left| \frac{\mathcal{M}}{e^4} \right|^2 \,. \tag{2.45}$$

The Feynman amplitude $\mathcal{M}$ depends on the polarization states of all incoming and outgoing particles. Cross-section shown in Eq. (2.45) can be therefore used to define all possible observables, like unpolarized cross-sections, asymmetries for various polarization states of beam and target, etc. If the target is polarized transversely, cross-section becomes dependent also on the angle φ_S illustrated in Fig. 2.2. In such case, one should consider an eight-fold differential cross-section:

$$\frac{d^8\sigma}{dx_B dQ^2 dQ'^2 d|t| d\phi d\Omega_\ell d\varphi_S} = \frac{\alpha_{\rm EM}^4}{16(2\pi)^3} \frac{x_B y^2}{Q^4\sqrt{1+\varepsilon^2}} \left|\frac{\mathcal{M}}{e^4}\right|^2 \times \frac{1}{2\pi}\,. \tag{2.46}$$

The amplitude receives contributions from all subprocesses (and their crossed partners) depicted in Fig. 2.1:

$$i\mathcal{M} = i\mathcal{M}_{\rm DDVCS} + i\mathcal{M}_{\rm BH1} + i\mathcal{M}_{\rm BH1X} + i\mathcal{M}_{\rm BH2} + i\mathcal{M}_{\rm BH2X}\,. \tag{2.47}$$

In the following sections we will separately evaluate each subamplitude. For this purpose we will employ the techniques developed by R. Kleiss and W. J. Stirling (KS) in Refs. [77, 78].

2.4.1 Kleiss-Stirling techniques

Usual computation of cross-sections implies dealing with traces of Dirac-gamma matrices that in turn renders complicated and lengthy sets of products of the momenta involved in the process. These issues become more problematic the larger the number of scattered particles is. For this reason, we turn to the KS techniques that allow for the reduction of amplitudes to complex scalars removing all spinors and gamma matrices. Actually, these scalars are nothing but complex numbers made out of components of momenta that, in practical computations, means to work with components of arrays. These objects are perfectly suited for computer applications, hence for implementation in the PARTONS platform.

The first step is to define a massless spinor basis $\{u(\kappa_0, \pm)\}$ for lightlike momentum $\kappa_0^2 = 0$ and helicity $\pm$. The negative-helicity state is defined up to a complex phase by the relation

$$\bar{u}(\kappa_0, -)u(\kappa_0, -) = \omega_- \not{\kappa}_0\,, \quad \omega_\lambda = \frac{1}{2}(1 + \lambda\gamma^5)\,. \tag{2.48}$$

Defining a spacelike vector $\kappa_1^2 = -1$ such that $\kappa_0\kappa_1 = 0$, one can prove that the positive-helicity state is given by

$$u(\kappa_0, +) = \not{\kappa}_1 u(\kappa_0, -)\,. \tag{2.49}$$

For any spinor associated to a massless fermion of momentum P, imposing Dirac equation, the projection relation

$$u(P, \lambda)\bar{u}(P, \lambda) = \omega_\lambda \not{P}\,, \quad \lambda = \pm\,, \tag{2.50}$$

and using the spinor basis above, one can find that the helicity states of such spinor can be written as

$$u(P, \lambda) = \frac{\not{P} u(\kappa_0, -\lambda)}{\sqrt{2P\kappa_0}}\,. \tag{2.51}$$

The only restriction to this formula is $P\kappa_0 \neq 0$ and, for computer purposes, not extremely small. In our analysis, an adequate choice for vector κ_0, κ_1 turns out to be

$$\kappa_0^\mu = (1, 1, 0, 0)\,, \quad \kappa_1^\mu = (0, 1, 0, 0)\,. \tag{2.52}$$

Once we know how to write spinors for lightlike momenta (2.51), one can define two scalars that will be at the core of our computation:

$$s(a,b) = \bar{u}(a,+)u(b,-) = -s(b,a)\,, \tag{2.53}$$
$$t(a,b) = \bar{u}(a,-)u(b,+) = [s(b,a)]^*\,, \tag{2.54}$$

where a and b are lightlike vectors.

In fact, $2ab = |s(a,b)|^2$ which means that all the products of momenta that would appear in the usual computation of the cross-section by means of traces of gamma matrices are concealed in these scalars. Moreover, for the choice (2.52), $s(a,b)$ acquires the simple form

$$s(a,b) = (a^2 + ia^3)\sqrt{\frac{b^0 - b^1}{a^0 - a^1}} - (a \leftrightarrow b)\,. \tag{2.55}$$

This completes the description of massless spinors. For fermions with momentum[2] p and non-zero mass m ($p^2 = m^2$), the helicity states are given in terms of any lightlike vector κ as

$$u(p,\lambda) = \frac{(\not{p} + m)u(\kappa,-)}{\sqrt{2p\kappa}}\,. \tag{2.56}$$

This form of the spinor satisfies both the Dirac equation and projection relation (2.49) so that

$$u(p,\pm)\bar{u}(p,\pm) = \frac{1}{2}(1 \pm \gamma^5 \not{s})(\not{p} + m)\,, \tag{2.57}$$

where

$$s^\mu = p^\mu/m - m\kappa^\mu/(p\kappa) \tag{2.58}$$

is the spin vector. It fulfills $s^2 = -1$ and $sp = 0$ so that Eq. (2.56) indeed describes a helicity state. Changing m by $-m$ in Eq. (2.56) renders the spinors for antiparticles $v(p,\pm)$.

Equation (2.56) can be written in terms of massless spinors of different helicities by finding two lightlike momenta κ, κ' such that $p = \kappa' + \kappa$:

$$u(p,+) = \frac{s(\kappa',\kappa)}{m}u(\kappa',+) + u(\kappa,-)\,, \tag{2.59}$$
$$u(p,-) = \frac{t(\kappa',\kappa)}{m}u(\kappa',-) + u(\kappa,+)\,, \tag{2.60}$$
$$v(p,+) = \frac{s(\kappa',\kappa)}{m}u(\kappa',+) - u(\kappa,-)\,, \tag{2.61}$$
$$v(p,-) = \frac{t(\kappa',\kappa)}{m}u(\kappa',-) - u(\kappa,+)\,. \tag{2.62}$$

As a consequence of the splitting $p = \kappa' + \kappa$, the spin vector simplifies to

$$s^\mu = (\kappa'^\mu - \kappa^\mu)/m\,. \tag{2.63}$$

[2]Here, p is a general timelike momentum, not necessarily equal to the proton momentum.

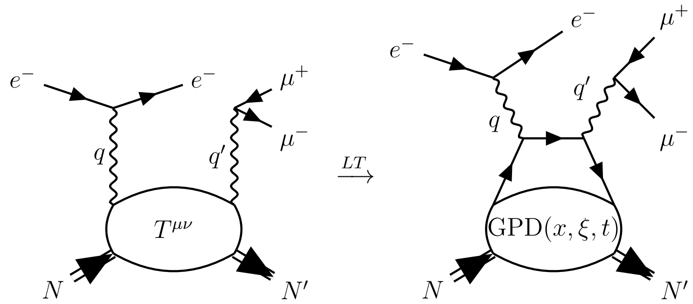

Fig. 2.3. General DDVCS diagram in terms of the Compton tensor $T^{\mu\nu}$ and its leading-twist (LT) approximation. The crossed diagram is not shown.

To conclude this section we introduce a couple of functions for later reference: the contraction of two currents

$$\begin{aligned} f(\lambda, k_0, k_1; \lambda', k_2, k_3) =& \bar{u}(k_0,\lambda)\gamma^\mu u(k_1,\lambda)\bar{u}(k_2,\lambda')\gamma_\mu u(k_3,\lambda') \\ =& 2[s(k_2,k_1)t(k_0,k_3)\delta_{\lambda-}\delta_{\lambda'+} + t(k_2,k_1)s(k_0,k_3)\delta_{\lambda+}\delta_{\lambda'-} \\ &+ s(k_2,k_0)t(k_1,k_3)\delta_{\lambda+}\delta_{\lambda'+} + t(k_2,k_0)s(k_1,k_3)\delta_{\lambda-}\delta_{\lambda'-}] , \end{aligned} \tag{2.64}$$

and the contraction of a current with a lightlike vector n

$$\begin{aligned} g(s, \ell, n, k) =& \bar{u}(\ell, s) \not{n} u(k, s) \\ =& \delta_{s+}s(\ell, n)t(n,k) + \delta_{s-}t(\ell,n)s(n,k) . \end{aligned} \tag{2.65}$$

For Eq. (2.64) we used the Chisholm identity

$$\bar{u}(k_0,\lambda)\gamma^\mu u(k_1,\lambda)(\gamma_\mu)_{ij} = 2u_i(k_1,\lambda)\bar{u}_j(k_0,\lambda) + 2u_i(k_0,-\lambda)\bar{u}_j(k_1,-\lambda) , \tag{2.66}$$

where i, j are spinor indices. For Eq. (2.65) we employed the relation

$$\not{n} = \sum_{\lambda=\pm} u(n,\lambda)\bar{u}(n,\lambda) , \tag{2.67}$$

which holds true as long as n is a lightlike vector.

The KS techniques includes methods for describing massless and massive bosons which we do not need as such particles are not part of the initial or final states in DDVCS. For further reading on the KS treatment of bosons cf. [77, 78].

2.4.2 DDVCS amplitude

The DDVCS contribution is a type of Compton scattering where the nucleon absorbs and emits virtual photons, thereby it is the only one related to the internal structure of the proton, i.e. to the GPDs. Its amplitude reads:

$$i\mathcal{M}_{\text{DDVCS}} = \frac{ie^4 \bar{u}(\ell_-, s_\ell)\gamma_\mu v(\ell_+, s_\ell)\bar{u}(k', s)\gamma_\nu u(k,s)}{(q^2 + i0)(q'^2 + i0)} T^{\mu\nu}_{s_2 s_1} , \tag{2.68}$$

where the Compton tensor $T^{\mu\nu}_{s_2 s_1}$ is given by Eq. (1.60), $s_\ell, s = \pm$ are the helicities of the muon and electron and $s_2, s_1 = \pm$ stand for hadron's helicity in the final and initial state, respectively.

To determine the leading terms in the Björken regime, this is the leading twist (LT) approximation of the Compton tensor, we make use of the light-cone coordinates once again. In the LT, the generalized Björken variable (1.92) simplifies to

$$\rho \simeq \xi \frac{Q^2 - Q'^2}{Q^2 + Q'^2}, \tag{2.69}$$

which is also the LT limit of the quotient $-\bar{q}^2/(2\bar{p}\bar{q})$. Hence, if we span the longitudinal plane by the vectors $\bar{p} = (p + p')/2$ and $\bar{q} = (q + q')/2$, then we can build two lightlike vectors n and n' such that $nn' = 1$ as combinations of $\bar{p}$ and $\bar{q}$:

$$n^\mu = \frac{1}{\bar{p}\bar{q}\tau}\bar{q}^\mu - \frac{1-\tau}{2\bar{p}\bar{q}\rho\delta^2\tau}\bar{p}^\mu, \tag{2.70}$$

$$n'^\mu = -\frac{\rho\delta^2}{\tau}\bar{q}^\mu + \frac{1+\tau}{2\tau}\bar{p}^\mu, \tag{2.71}$$

where

$$\tau = \sqrt{1 + 4\rho^2\delta^2} \quad \text{and} \quad \delta^2 = \frac{M^2 - t/4}{2\bar{p}\bar{q}\rho}. \tag{2.72}$$

The vectors n and n' allow for the decomposition of any four-vector as

$$v^\mu = v^- n^\mu + v^+ n'^\mu + v_\perp^\mu. \tag{2.73}$$

Taking into account that

$$\rho^2\delta^2 = O\left(\frac{M^2}{Q^2 + Q'^2}, \frac{|t|}{Q^2 + Q'^2}\right), \tag{2.74}$$

it follows that powers of $\rho^2\delta^2$ generate kinematic higher-twist corrections and to the LT we can drop them. As a result, expanding the decomposition of the Compton tensor introduced in Ref. [42] (which is valid up to kinematic twist-3 accuracy) by means of this variable and isolating the terms that survives the limit $\rho^2\delta^2 \to 0$, we are left with:

$$\begin{aligned} T^{\mu\nu}_{s_2 s_1} = {} & T^{(V)\mu\nu}\bar{u}(p', s_2)\left[(\mathcal{H} + \mathcal{E})\not{n} - \frac{\mathcal{E}}{M}\bar{p}^+\right]u(p, s_1) \\ & + T^{(A)\mu\nu}\bar{u}(p', s_2)\left[\widetilde{\mathcal{H}}\not{n} + \frac{\widetilde{\mathcal{E}}}{2M}\Delta^+\right]\gamma^5 u(p, s_1). \end{aligned} \tag{2.75}$$

Actually, this expression can be read out from Eq. (1.77). As already introduced in Ch. 1, Compton form factors (CFFs) are defined as:

$$(\mathcal{H}, \mathcal{E})(\rho, \xi, t) = \sum_{f=\{u,d,s\}} \int_{-1}^{1} dx\, C_f^{(-)}(x, \rho)(H_f, E_f)(x, \xi, t), \tag{2.76}$$

$$(\widetilde{\mathcal{H}}, \widetilde{\mathcal{E}})(\rho, \xi, t) = \sum_{f=\{u,d,s\}} \int_{-1}^{1} dx\, C_f^{(+)}(x, \rho)(\widetilde{H}_f, \widetilde{E}_f)(x, \xi, t), \tag{2.77}$$

where $C_f^{(\pm)}$ are hard scattering coefficient functions, which at LO and LT read:

$$C_f^{(\pm)}(x, \rho) = \left(\frac{e_f}{e}\right)^2 \left(\frac{1}{\rho - x - i0} \pm \frac{1}{\rho + x - i0}\right), \tag{2.78}$$

and where $H, E, \widetilde{H}$ and $\widetilde{E}$ are GPDs related to quark correlators as in Eqs. (1.78) and (1.79). Note that $\bar{p}^+ = 1$ so we might as well define a vector

$$\begin{aligned} \mathcal{J}^{+}_{s_2 s_1} &= \sum_{f=\{u,d,s\}} \int_{-1}^{1} dx\, C_f^{(-)}(x,\rho) \int \frac{d\lambda}{2\pi}\, e^{-i\lambda x} \langle p', s_2 | \bar{\mathfrak{q}}_f(\lambda n/2) \not{n}\, \mathfrak{q}_f(-\lambda n/2) | p, s_1 \rangle \\ &= \bar{u}(p', s_2) \left[(\mathcal{H} + \mathcal{E}) \not{n} - \frac{\mathcal{E}}{M} \bar{p}^+ \right] u(p, s_1) , \end{aligned} \tag{2.79}$$

and an axial-vector

$$\begin{aligned} \mathcal{J}^{(5)+}_{s_2 s_1} &= \sum_{f=\{u,d,s\}} \int_{-1}^{1} dx\, C_f^{(+)}(x,\rho) \int \frac{d\lambda}{2\pi}\, e^{-i\lambda x} \langle p', s_2 | \bar{\mathfrak{q}}_f(\lambda n/2) \not{n} \gamma^5 \mathfrak{q}_f(-\lambda n/2) | p, s_1 \rangle \\ &= \bar{u}(p', s_2) \left[\widetilde{\mathcal{H}} \not{n} + \frac{\widetilde{\mathcal{E}}}{2M} \Delta^+ \right] \gamma^5 u(p, s_1) , \end{aligned} \tag{2.80}$$

hadron currents projected with n. Finally, the dominant Lorentz components have been isolated and concealed in the following tensors:

$$T^{(V)\mu\nu} = -\frac{1}{2}(g^{\mu\nu} - n^\mu n'^\nu - n^\nu n'^\mu) \equiv -\frac{1}{2} g_\perp^{\mu\nu} , \tag{2.81}$$

$$T^{(A)\mu\nu} = -\frac{i}{2} \epsilon^{\mu\nu}{}_{\rho\sigma} n^\rho n'^\sigma \equiv -\frac{i}{2} \epsilon_\perp^{\mu\nu} , \tag{2.82}$$

where $\epsilon^{0123} = +1$ and, as anticipated in Ch. 1, we can define a vector and an axial components of the DDVCS amplitude:

$$i\mathcal{M}_{\text{DDVCS}} = \frac{-ie^4}{(Q^2 - i0)(Q'^2 + i0)} \left(i\mathcal{M}^{(V)}_{\text{DDVCS}} + i\mathcal{M}^{(A)}_{\text{DDVCS}} \right) . \tag{2.83}$$

Here, the first term (the vector contribution) corresponds to $T^{(V)}$, while the second one (the axial contribution) to $T^{(A)}$. In what follows we study these two contributions separately.

Vector contribution to the DDVCS amplitude

Up to photon propagators and factors ie^4, the vector amplitude may be written as:

$$i\mathcal{M}^{(V)}_{\text{DDVCS}} = -\frac{g_\perp^{\mu\nu}}{2} \bar{u}(\ell_-, s_\ell) \gamma_\mu v(\ell_+, s_\ell) \bar{u}(k', s) \gamma_\nu u(k, s) \mathcal{J}^{+}_{s_2 s_1} , \tag{2.84}$$

where $s_\ell, s = \pm$ stand for muon's and electron's helicities, respectively. The current $\mathcal{J}^+$ is given by Eq. (2.79) and can be further decomposed into:

$$\mathcal{J}^{+}_{s_2 s_1} = (\mathcal{H} + \mathcal{E}) \mathcal{J}^{(1)+}_{s_2 s_1} - \frac{\mathcal{E}}{M} \mathcal{J}^{(2)}_{s_2 s_1} , \tag{2.85}$$

where

$$\mathcal{J}^{(1)\mu}_{s_2 s_1} = \bar{u}(p', s_2) \gamma^\mu u(p, s_1) , \quad \mathcal{J}^{(2)}_{s_2 s_1} = \bar{u}(p', s_2) u(p, s_1) . \tag{2.86}$$

To use the KS methods for massive spinors, hadron momenta p and p' have to be decomposed by means of auxiliary lightlike vectors, this is $p = r_1 + r_2$ and $p' = r_1' + r_2'$.

In App. B a possible choice for $\{r_1, r_2\}$ and $\{r'_1, r'_2\}$ is given, although the formulation presented here is independent of the choice of these lightlike vectors.

The decomposition of nucleon spinors into massles ones is given in Eqs. (2.59) and (2.60) which can be used to compute $\mathcal{J}^{(1)\mu}$:

$$\mathcal{J}^{(1)\mu}_{s_2 s_1} = Y_{s_2 s_1} \bar{u}(r'_{s_2}, +)\gamma^\mu u(r_{s_1}, +) + Z_{s_2 s_1} \bar{u}(r'_{-s_2}, -)\gamma^\mu u(r_{-s_1}, -) \,, \tag{2.87}$$

where $r'_{s_2} = r'_1 \delta_{s_2+} + r'_2 \delta_{s_2-}$ and $r_{s_1} = r_1 \delta_{s_1+} + r_2 \delta_{s_1-}$. After introducing the scalars from Eqs. (2.53) and (2.54), phases[3] Y, Z read:

$$Y_{s_2 s_1} = \delta_{s_2+}\delta_{s_1+} \frac{t(r'_2, r'_1) s(r_1, r_2)}{M^2} + \delta_{s_2+}\delta_{s_1-} \frac{t(r'_2, r'_1)}{M} + \delta_{s_2-}\delta_{s_1+} \frac{s(r_1, r_2)}{M} + \delta_{s_2-}\delta_{s_1-} \tag{2.88}$$

and

$$Z_{s_2 s_1} = \delta_{s_2-}\delta_{s_1-} \frac{s(r'_2, r'_1) t(r_1, r_2)}{M^2} + \delta_{s_2-}\delta_{s_1+} \frac{s(r'_2, r'_1)}{M} + \delta_{s_2+}\delta_{s_1-} \frac{t(r_1, r_2)}{M} + \delta_{s_2+}\delta_{s_1+} \,. \tag{2.89}$$

A similar calculation for $\mathcal{J}^{(2)}_{s_2 s_1}$ yields:

$$\begin{aligned} \mathcal{J}^{(2)}_{s_2 s_1} = \; & \delta_{s_2+}\delta_{s_1+} \left[\frac{t(r'_2, r'_1) s(r'_1, r_2)}{M} + \frac{t(r'_2, r_1) s(r_1, r_2)}{M} \right] \\ & + \delta_{s_2+}\delta_{s_1-} \left[\frac{t(r'_2, r'_1) t(r_1, r_2) s(r'_1, r_1)}{M^2} + t(r'_2, r_2) \right] \\ & + \delta_{s_2-}\delta_{s_1+} \left[\frac{s(r'_2, r'_1) s(r_1, r_2) t(r'_1, r_1)}{M^2} + s(r'_2, r_2) \right] \\ & + \delta_{s_2-}\delta_{s_1-} \left[\frac{s(r'_2, r'_1) t(r'_1, r_2)}{M} + \frac{s(r'_2, r_1) t(r_1, r_2)}{M} \right] . \end{aligned} \tag{2.90}$$

By contracting Eq. (2.87) with vector n and employing g function as given in Eq. (2.65) we arrive to:

$$\mathcal{J}^{(1)+}_{s_2 s_1} = \mathcal{J}^{(1)\mu}_{s_2 s_1} n_\mu = Y_{s_2 s_1} g(+, r'_{s_2}, n, r_{s_1}) + Z_{s_2 s_1} g(-, r'_{-s_2}, n, r_{-s_1}) \,. \tag{2.91}$$

Finally, the vector contribution to the DDVCS amplitude is:

$$\begin{aligned} & i\mathcal{M}^{(V)}_{\text{DDVCS}} = \\ & = \frac{-1}{2} \left[f(s_\ell, \ell_-, \ell_+; s, k', k) - g(s_\ell, \ell_-, n', \ell_+) g(s, k', n, k) - g(s_\ell, \ell_-, n, \ell_+) g(s, k', n', k) \right] \\ & \times \left[(\mathcal{H} + \mathcal{E}) [Y_{s_2 s_1} g(+, r'_{s_2}, n, r_{s_1}) + Z_{s_2 s_1} g(-, r'_{-s_2}, n, r_{-s_1})] - \frac{\mathcal{E}}{M} \mathcal{J}^{(2)}_{s_2 s_1} \right] . \end{aligned} \tag{2.92}$$

Each of the terms in this vector amplitude has a clear physical meaning as contraction between lepton currents and momenta weighted by the parton content of the proton represented by the CFFs $\mathcal{H}$ and $\mathcal{E}$.

[3] Y and Z have unit modulus as $|s(r_1, r_2)|^2 = 2 r_1 r_2 = M^2$. Likewise, for $s \leftrightarrow t$ and/or $r_{1,2} \leftrightarrow r'_{1,2}$.

Axial contribution to the DDVCS amplitude

The axial amplitude may be written, up to photon propagators and factors ie^4, as:

$$i\mathcal{M}^{(A)}_{\text{DDVCS}} = \frac{-i\epsilon_\perp^{\mu\nu}}{2}\bar{u}(\ell_-,s_\ell)\gamma_\mu v(\ell_+,s_\ell)\bar{u}(k',s)\gamma_\nu u(k,s)\mathcal{J}^{(5)+}_{s_2s_1}\,, \tag{2.93}$$

where $\mathcal{J}^{(5)+}$ was defined in Eq. (2.80). We distinguish $\mathcal{J}^{(1,5)+}$ and $\mathcal{J}^{(2,5)+}$:

$$\mathcal{J}^{(5)+}_{s_2s_1} = \widetilde{\mathcal{H}}\mathcal{J}^{(1,5)+}_{s_2s_1} + \widetilde{\mathcal{E}}\frac{\Delta^+}{2M}\mathcal{J}^{(2,5)+}_{s_2s_1}\,, \tag{2.94}$$

such that

$$\mathcal{J}^{(1,5)+}_{s_2s_1} = \bar{u}(p',s_2)\not{n}\gamma^5 u(p,s_1)\,, \tag{2.95}$$

$$\mathcal{J}^{(2,5)+}_{s_2s_1} = \bar{u}(p',s_2)\gamma^5 u(p,s_1)\,. \tag{2.96}$$

These bilinears are the axial partners of those in Eq. (2.86). Similarly to the vector case, we can express the currents above by means of the scalar functions (2.53) and (2.54):

$$\begin{aligned}\mathcal{J}^{(1,5)+}_{s_2s_1} &= \delta_{s_2+}\delta_{s_1+}\left[\frac{t(r_2',r_1')s(r_1,r_2)t(n,r_1)s(r_1',n)}{M^2} - s(n,r_2)t(r_2',n)\right]\\ &- \delta_{s_2-}\delta_{s_1-}\left[\frac{s(r_2',r_1')t(r_1,r_2)s(n,r_1)t(r_1',n)}{M^2} - t(n,r_2)s(r_2',n)\right]\\ &+ \delta_{s_2+}\delta_{s_1-}\frac{t(r_2',r_1')t(n,r_2)s(r_1',n) - t(r_1,r_2)s(n,r_1)t(r_2',n)}{M}\\ &- \delta_{s_2-}\delta_{s_1+}\frac{s(r_2',r_1')s(n,r_2)t(r_1',n) - s(r_1,r_2)t(n,r_1)s(r_2',n)}{M}\end{aligned} \tag{2.97}$$

and

$$\begin{aligned}\mathcal{J}^{(2,5)+}_{s_2s_1} &= \delta_{s_2+}\delta_{s_1+}\frac{s(r_1,r_2)t(r_2',r_1) - t(r_2',r_1')s(r_1',r_2)}{M}\\ &- \delta_{s_2-}\delta_{s_1-}\frac{t(r_1,r_2)s(r_2',r_1) - s(r_2',r_1')t(r_1',r_2)}{M}\\ &+ \delta_{s_2+}\delta_{s_1-}\left[t(r_2',r_2) - \frac{t(r_2',r_1')t(r_1,r_2)s(r_1',r_1)}{M^2}\right]\\ &- \delta_{s_2-}\delta_{s_1+}\left[s(r_2',r_2) - \frac{s(r_2',r_1')s(r_1,r_2)t(r_1',r_1)}{M^2}\right].\end{aligned} \tag{2.98}$$

Unlike the vector case, because of the $\epsilon_\perp$-structure in Eq. (2.93) we cannot contract the lepton currents, so we need to compute them explicitly. Using the spinor representation (2.51) for the massless lepton current of the muon-antimuon pair we have:

$$\begin{aligned}j_\mu(s_\ell,\ell_-,\ell_+) &= \bar{u}(\ell_-,s_\ell)\gamma_\mu v(\ell_+,s_\ell)\\ &= \frac{1}{2N_{\ell_-\ell_+}}\text{tr}\left\{\not{\ell}_-\gamma_\mu\not{\ell}_+ u(\kappa_0,-s_\ell)\bar{u}(\kappa_0,-s_\ell)\right\}\\ &= \frac{1}{2N_{\ell_-\ell_+}}\text{tr}\left\{\not{\ell}_-\gamma_\mu\not{\ell}_+\omega_{-s_\ell}\not{\kappa}_0\right\}\end{aligned}$$

$$= \frac{1}{N_{\ell_-\ell_+}}\{\ell_{-,\mu}(\ell_+\kappa_0) + \ell_{+,\mu}(\ell_-\kappa_0) - \kappa_{0,\mu}Q'^2/2 + is_\ell\epsilon_{\mu\alpha\beta\gamma}\ell_-^\alpha\ell_+^\beta\kappa_0^\gamma\}\,, \tag{2.99}$$

where $N_{\ell_-\ell_+} = \sqrt{(\ell_-\kappa_0)(\ell_+\kappa_0)}$ and $\epsilon_{0123} = -1$.

Likewise, for the electron current with a normalization factor $N_{k'k} = \sqrt{(k'\kappa_0)(k\kappa_0)}$:

$$j_\mu(s,k',k) = \frac{1}{N_{k'k}}\{k'_\mu(k\kappa_0) + k_\mu(k'\kappa_0) - \kappa_{0,\mu}Q^2/2 + is\epsilon_{\mu\alpha\beta\gamma}k'^\alpha k^\beta\kappa_0^\gamma\}\,. \tag{2.100}$$

Finally, the axial contribution is given by:

$$i\mathcal{M}^{(A)}_{\text{DDVCS}} = \frac{-i}{2}\epsilon_\perp^{\mu\nu}j_\mu(s_\ell,\ell_-,\ell_+)j_\nu(s,k',k)\left[\widetilde{\mathcal{H}}\mathcal{J}^{(1,5)+}_{s_2s_1} + \widetilde{\mathcal{E}}\frac{\Delta^+}{2M}\mathcal{J}^{(2,5)+}_{s_2s_1}\right]\,. \tag{2.101}$$

The physical interpretation of this formula is straightforward: interaction of lepton and quark currents weighted by the CFFs $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{E}}$, which describe the parton content of the proton.

Violation of the gauge invariance due to a truncation of the twist expansion of the Compton tensor is a problem discussed in papers like [79, 80] and more recently in [67, 68]. From the decomposition of the Compton tensor given by Eqs. (2.81) and (2.82) and the chosen longitudinal plane spanned by n (2.70) and n' (2.71) we realize that the photons carry transverse momenta:

$$q_\perp^\mu = -q_\perp'^\mu \propto \Delta_\perp^\mu \neq 0\,. \tag{2.102}$$

As a consequence, the electromagnetic gauge invariance is violated by terms of order $O(\Delta_\perp/\sqrt{2\bar{p}\bar{q}})$, which are twist-3 effects:

$$q'_\mu T^{(V)\mu\nu} \propto \Delta_\perp^\nu \neq 0\,. \tag{2.103}$$

In Ref. [81] it was proven that by going to twist-3, violation is of order twist-4 and so on. Despite restoring gauge invariance is possible twist-by-twist, the existence of gauge symmetry-breaking terms affects predictions at LT. There are two ways to deal with it: 1) Lorentz transformation to a reference frame where photons do not carry perpendicular components, or 2) evaluation of the hard part, which includes the Lorentz structures of the Compton tensor (2.81) and (2.82), at $t = t_0$ (2.24). Choosing to compute at t_0 ensures that higher-twist corrections proportional to $\Delta_\perp^\mu$ vanish, avoiding violation of the gauge invariance. This evaluation is consistent with the collinear factorization that is at the core of the GPD description, where transverse dynamics are neglected. In this chapter we opt for this second option. As a consequence, the longitudinal vectors n and n' are also evaluated at $t = t_0$ in the hard part.

2.4.3 The first Bethe-Heitler amplitude

Now we describe the amplitude of BH1 and its crossed partner BH1X, both depicted in Fig. 2.4.

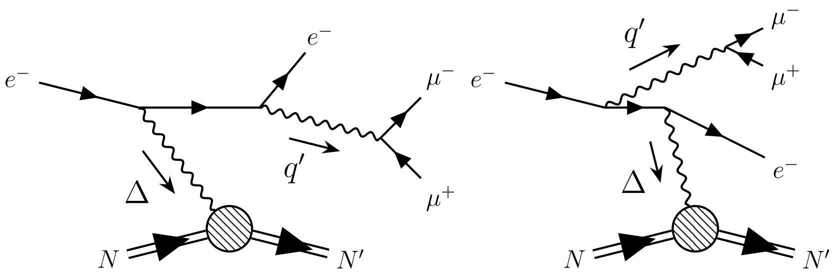

Fig. 2.4. Diagrams for Bethe-Heitler 1 (BH1) and its crossed partner (BH1X) for electroproduction of muon pairs.

The amplitude of BH1 reads:

$$i\mathcal{M}_{\rm BH1} = \frac{ie^4\bar{u}(\ell_-, s_\ell)\gamma^\beta v(\ell_+, s_\ell)\bar{u}(k', s)\gamma_\beta(\not{k} - \not{\Delta})\gamma_\alpha u(k, s)J^\alpha_{s_2 s_1}}{(Q'^2 + i0)(t + i0)((k - \Delta)^2 + i0)}, \tag{2.104}$$

where the electromagnetic hadronic current is parameterized by means of Dirac, $F_1(t)$, and Pauli, $F_2(t)$, elastic form factors (EFFs):

$$J^\alpha_{s_2 s_1} = \langle p', s_2| \sum_f \frac{e_f}{e} \bar{\mathfrak{q}}_f(0)\gamma^\alpha \mathfrak{q}_f(0)|p, s_1\rangle = \bar{u}(p', s_2)\left[(F_1 + F_2)\gamma^\alpha - \frac{F_2}{M}\bar{p}^\alpha\right]u(p, s_1). \tag{2.105}$$

The physical meaning of the EFFs was already discussed in Ch. 1 through DIS. In App. C we specify the parameterization used in our phenomenological studies and implemented in `PARTONS`.

The structure of this current is the same as that of $\mathcal{J}^+$, see Eq. (2.79). Therefore, up to CFF $\leftrightarrow$ EFF replacement, one may apply the decomposition (2.85):

$$J^\alpha_{s_2 s_1} = (F_1 + F_2)\mathcal{J}^{(1)\alpha}_{s_2 s_1} - \frac{F_2}{M}\bar{p}^\alpha \mathcal{J}^{(2)}_{s_2 s_1}, \tag{2.106}$$

where $\mathcal{J}^{(1)\alpha}$ and $\mathcal{J}^{(2)}$ are defined in Eqs. (2.87) and (2.90), respectively.

With the decomposition (2.106) the evaluation of $i\mathcal{M}_{\rm BH1}$ can be separated into two parts related to $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$:

$$i\mathcal{M}_{\rm BH1} = \frac{ie^4\left(i\mathcal{M}^{(1)}_{\rm BH1} + i\mathcal{M}^{(2)}_{\rm BH1}\right)}{(Q'^2 + i0)(t + i0)((k - \Delta)^2 + i0)}. \tag{2.107}$$

The first term in the numerator of Eq. (2.107), by means of the scalar function f introduced in Eq. (2.64), can be expressed as:

$$\begin{aligned} i\mathcal{M}^{(1)}_{\rm BH1} = (F_1 + F_2)\sum_L f(s_\ell, \ell_-, \ell_+; s, k', L)\Big(&Y_{s_2 s_1} f(s, L, k; +, r'_{s_2}, r_{s_1}) \\ &+ Z_{s_2 s_1} f(s, L, k; -, r'_{-s_2}, r_{-s_1})\Big), \end{aligned} \tag{2.108}$$

where $L \in \{k', \ell_-, \ell_+\}$.

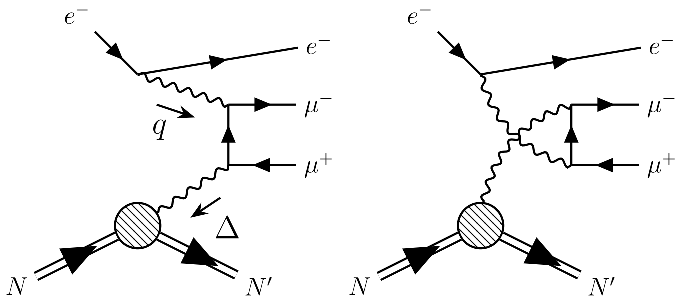

Fig. 2.5. Diagrams for Bethe-Heitler 2 (BH2) and its crossed partner (BH2X) for electroproduction of muon pairs.

The second term in Eq. (2.107), after expanding $\bar{p}$ in the sum of lightlike vectors $R \in \{r_1, r_2, r_1', r_2'\}$, has the following form:

$$i\mathcal{M}^{(2)}_{\mathrm{BH1}} = -\frac{F_2}{2M}\mathcal{J}^{(2)}_{s_2 s_1}\sum_{L,R} f(s_\ell, \ell_-, \ell_+; s, k', L) g(s, L, R, k) \,. \tag{2.109}$$

The amplitude of crossed BH1 reads:

$$i\mathcal{M}_{\mathrm{BH1X}} = \frac{ie^4 \bar{u}(\ell_-, s_\ell)\gamma^\beta v(\ell_+, s_\ell)\bar{u}(k', s)\gamma_\alpha(\not{k}' + \not{\Delta})\gamma_\beta u(k, s) J^\alpha_{s_2 s_1}}{(q'^2 + i0)(t + i0)((k' + \Delta)^2 + i0)} \,. \tag{2.110}$$

Analogously to Eqs. (2.107), (2.108) and (2.109):

$$i\mathcal{M}_{\mathrm{BH1X}} = \frac{ie^4\left(i\mathcal{M}^{(1)}_{\mathrm{BH1X}} + i\mathcal{M}^{(2)}_{\mathrm{BH1X}}\right)}{(Q'^2 + i0)(t + i0)((k' + \Delta)^2 + i0)} \,, \tag{2.111}$$

and

$$i\mathcal{M}^{(1)}_{\mathrm{BH1X}} = (F_1 + F_2)\sum_L \sigma(L) f(s_\ell, \ell_-, \ell_+; s, L, k')\Big(Y_{s_2 s_1} f(s, k', L; +, r'_{s_2}, r_{s_1}) + Z_{s_2 s_1} f(s, k', L; -, r'_{-s_2}, r_{-s_1})\Big) \,, \tag{2.112}$$

$$i\mathcal{M}^{(2)}_{\mathrm{BH1X}} = -\frac{F_2}{2M}\mathcal{J}^{(2)}_{s_2 s_1}\sum_{L,R}\sigma(L) f(s_\ell, \ell_-, \ell_+; s, L, k) g(s, k', R, L) \,, \tag{2.113}$$

where $L \in \{k, \ell_-, \ell_+\}$, $R \in \{r_1, r_2, r_1', r_2'\}$ and $\sigma(k) = +1$, $\sigma(\ell_-) = \sigma(\ell_+) = -1$.

2.4.4 The second Bethe-Heitler amplitude

The Feynman diagrams for BH2 and its crossed partner are shown in Fig. 2.5. The amplitude of the former reads:

$$i\mathcal{M}_{\mathrm{BH2}} = \frac{ie^4 \bar{u}(k', s)\gamma^\beta u(k, s)\bar{u}(\ell_-, s_\ell)\gamma_\beta(\not{k} - \not{k}' - \not{\ell}_-)\gamma_\alpha v(\ell_+, s_\ell) J^\alpha_{s_2 s_1}}{(Q^2 - i0)(t + i0)((q - \ell_-)^2 + i0)} \,, \tag{2.114}$$

and, as for BH1, can be split into two terms corresponding to elements of Eq. (2.106) as:

$$i\mathcal{M}_{\text{BH2}} = \frac{ie^4\left(i\mathcal{M}^{(1)}_{\text{BH2}} + i\mathcal{M}^{(2)}_{\text{BH2}}\right)}{(Q^2 - i0)(t+i0)((q-\ell_-)^2 + i0)}\,. \tag{2.115}$$

The same steps as presented in the previous sections lead to:

$$i\mathcal{M}^{(1)}_{\text{BH2}} = (F_1+F_2)\sum_L \sigma(L) f(s_\ell,\ell_-,L;s,k',k)\Big(Y_{s_2s_1} f(s_\ell,L,\ell_+;+,r'_{s_2},r_{s_1}) + Z_{s_2s_1} f(s_\ell,L,\ell_+;-,r'_{-s_2},r_{-s_1})\Big)\,, \tag{2.116}$$

$$i\mathcal{M}^{(2)}_{\text{BH2}} = \frac{-F_2}{2M}\mathcal{J}^{(2)}_{s_2s_1}\sum_{L,R}\sigma(L) f(s_\ell,\ell_-,L;s,k',k) g(s_\ell,L,R,\ell_+)\,, \tag{2.117}$$

where $L \in \{k,k',\ell_-\}$, $R \in \{r_1,r_2,r'_1,r'_2\}$, $\sigma(k) = +1$ and $\sigma(k') = \sigma(\ell_-) = -1$.

The amplitude for the crossed partner of BH2 is given by:

$$i\mathcal{M}_{\text{BH2X}} = \frac{ie^4\bar{u}(\ell_-,s_\ell)\gamma_\alpha(\not{k} - \not{k}' - \not{\ell}_+)\gamma_\beta v(\ell_+,s_\ell)\bar{u}(k',s)\gamma^\beta u(k,s) J^\alpha_{s_2s_1}}{(Q'^2 + i0)(t+i0)((q-\ell_+)^2 + i0)}\,. \tag{2.118}$$

It can be expressed by:

$$i\mathcal{M}_{\text{BH2X}} = \frac{-ie^4\left(i\mathcal{M}^{(1)}_{\text{BH2X}} + i\mathcal{M}^{(2)}_{\text{BH2X}}\right)}{(Q^2 - i0)(t+i0)((q-\ell_+)^2 + i0)}\,, \tag{2.119}$$

for which

$$i\mathcal{M}^{(1)}_{\text{BH2X}} = (F_1+F_2)\sum_L \sigma(L) f(s_\ell,L,\ell_+;s,k',k)\Big(Y_{s_2s_1} f(s_\ell,\ell_-,L;+,r'_{s_2},r_{s_1}) + Z_{s_2s_1} f(s_\ell,\ell_-,L;-,r'_{-s_2},r_{-s_1})\Big)\,, \tag{2.120}$$

$$i\mathcal{M}^{(2)}_{\text{BH2X}} = -\frac{F_2}{2M}\mathcal{J}^{(2)}_{s_2s_1}\sum_{L,R}\sigma(L) f(s_\ell,L,\ell_+;s,k',k) g(s_\ell,\ell_-,R,L)\,, \tag{2.121}$$

with $L \in \{k,k',\ell_+\}$, $R \in \{r_1,r_2,r'_1,r'_2\}$ and $\sigma(k) = +1, \sigma(k') = \sigma(\ell_+) = -1$.

2.4.5 Polarized target

Although the observables that we will present in Sect. 2.6 consider an unpolarized target, for completeness we describe in this section how to address the case of a polarized target within the Kleiss-Stirling approach. Thus far, hadron polarization denoted with index s_1 corresponds to the values $\pm$ for helicity with respect to the three-vector component of, vid. Eq. (2.63),

$$s^\mu = (r_1^\mu - r_2^\mu)/M\,, \tag{2.122}$$

with M the target mass and r_1, r_2 two lightlike vectors, provided that the hadron momentum can be written as $p = r_1 + r_2$.

With respect to TRF-II axes and the choice of r_1 and r_2 in App. B, s reads:

$$s^\mu = (0,\hat{z})\,, \quad \hat{z} = (0,0,1)\,. \tag{2.123}$$

Therefore, $\vec{s} = \hat{z}$ is parallel to the outgoing photon three-momentum $\vec{q}\,'$. The relation between the quantization of helicity in direction $\hat{z}$ (denoted by $s_1 = \pm$) and in another direction defined by the three-vector[4] $\vec{S} = (\sin\theta_S \cos\phi_S, \sin\theta_S \sin\phi_S, \cos\theta_S)$ is:

$$|h_1 = +\rangle = \cos(\theta_S/2)|s_1 = +\rangle + e^{i\phi_S}\sin(\theta_S/2)|s_1 = -\rangle\,, \tag{2.124}$$

$$|h_1 = -\rangle = -e^{-i\phi_S}\sin(\theta_S/2)|s_1 = +\rangle + \cos(\theta_S/2)|s_1 = -\rangle\,. \tag{2.125}$$

Introducing these spin states in the Compton tensor (1.60) and the electromagnetic hadron current (2.105) is equivalent to relate spinors $u'(p, h_1)$ to $u(p, s_1)$ via:

$$u'(p, h_1) = F_{h_1+}u(p, +) + F_{h_1-}u(p, -)\,, \tag{2.126}$$

where matrix F has been defined in accordance to (2.124) and (2.125) as

$$F = \begin{pmatrix} \cos\frac{\theta_S}{2} & e^{i\phi_S}\sin\frac{\theta_S}{2} \\ -e^{-i\phi_S}\sin\frac{\theta_S}{2} & \cos\frac{\theta_S}{2} \end{pmatrix} = \begin{pmatrix} F_{++} & F_{+-} \\ F_{-+} & F_{--} \end{pmatrix}\,. \tag{2.127}$$

Therefore, using (2.126) in Eqs. (2.79), (2.80) and (2.105), the correspondence between our current amplitudes, where target is polarized in direction $\hat{z}$ (index s_1), and the ones with a target polarized with respect to $\vec{S}$ (index h_1) reads:

$$i\mathcal{M}(s_2, h_1) = F_{h_1+}i\mathcal{M}(s_2, s_1 = +) + F_{h_1-}i\mathcal{M}(s_2, s_1 = -)\,. \tag{2.128}$$

We can orientate $\vec{S}$ in such a way that we define two types of target polarization: longitudinal ($\vec{S} \parallel \vec{k}$) and transverse ($\vec{S} \perp \vec{k}$) to the electron beam, which are detailed in the following.

Longitudinal polarization

We define longitudinal polarization as the polarization with respect to the electron beam $\vec{S} = \vec{k}/|\vec{k}|$. This one has been parameterized in Sect. 2.2. To build the matrix F we need angles θ_S and ϕ_S, which are the polar and azimuthal coordinates of the vector $\vec{k}$ with respect to TRF-II.

Hence, the polar angle is:

$$\hat{z}\vec{k}/|\vec{k}| = \cos\theta_S \Rightarrow \cos\theta_S = -s_e s_\gamma c_\phi + c_e c_\gamma\,, \tag{2.129}$$

whereas the azimuthal angle is:

$$\tan\phi_S = \frac{k^2}{k^1} = -\frac{s_e s_\phi}{s_e c_\gamma c_\phi + c_e s_\gamma}\,. \tag{2.130}$$

The signs of k^1 and k^2 inform us about the quadrant in which ϕ_S can be found, so it is fully determined by the above formula. Introducing these angles in Eqs. (2.127) and (2.128) we obtain the amplitude for positive ($h_1 = +$) and negative ($h_1 = -$) longitudinal polarization with respect to the electron beam.

[4] Angles ϕ_S and θ_S are the azimuthal and polar orientations of $\vec{S}$ with respect to TRF-II.

Transverse polarization

In the case of a transversely polarized target, we need a vector $\vec{S}$ perpendicular to the beam, this is perpendicular to $\vec{k}$. Such a vector is ambiguous as there are an infinite number of perpendicular vectors to a given one in three dimensions. To account for this ambiguity, we include the azimuthal angle φ_S (with respect to TRF-I) of the transverse vector $\vec{S}$ as an extra variable in the differential cross-section so we can ultimately integrate over the total of them, see Eq. (2.46).

For each $\vec{S}$ perpendicular to $\vec{k}$ with a particular azimuthal angle φ_S with respect to TRF-I, the corresponding polar angle ϑ_S is fixed by the orthogonality condition

$$\vec{S}\vec{k} = 0 \Rightarrow \tan\vartheta_S = \frac{s_e s_\gamma c_\phi - c_e c_\gamma}{\sin\varphi_S s_e s_\phi - \cos\varphi_S (s_e c_\gamma c_\phi + c_e s_\gamma)} . \tag{2.131}$$

As ϑ_S is a polar coordinate, it is in the range $(0, \pi)$ rad so that the sine is always positive. Consequently, with the sign of the tangent we can fully determine the quadrant ϑ_S is in and so its value given the azimuthal φ_S. With the Lorentz transformation (2.17) we can relate the azimuthal and polar angles ϕ_S, θ_S in TRF-II needed in the matrix F to those in TRF-I (φ_S, ϑ_S). Therefore, from Eq. (2.128) the transversely polarized amplitude is

$$i\mathcal{M}(s_2, \varphi_S) = F_{++}(\varphi_S) i\mathcal{M}(s_2, s_1 = +) + F_{+-}(\varphi_S) i\mathcal{M}(s_2, s_1 = -) , \tag{2.132}$$

where we stressed-out that the amplitude depends on φ_S through the TRF-II's ϕ_S and θ_S angles in the matrix F.

2.5 DVCS and TCS limits

As indicated earlier, DDVCS serves as a single framework that can be used to study DVCS and TCS. Since there is no data on DDVCS to compare with, we will use this feature to prove the validity of the formalism developed in Sect. 2.4. Recovering the DVCS cross-section requires:

1. Integrating over the muon pair, in other words to perform the integral over the solid angle Ω_ℓ of the muon-antimuon system. This removes the produced leptons of DDVCS.

2. Taking the $Q'^2 \to 0$ limit. This makes the photon real, as it should be for DVCS. Additionally, this implies the evaluation of the CFF for $\rho \to \xi$.

The resulting cross-section corresponds to that of DVCS up to a residual factor from the substitution of the $\gamma^*(q'^2 \neq 0)\mu^-\mu^+$ system by a single real photon $\gamma(q'^2 = 0)$. This factor accounts for the propagator of the virtual photon and its splitting on the $\mu^-\mu^+$ pair:

$$\int d\Omega_\ell \underbrace{\frac{d^7\sigma}{dx_B dQ^2 dQ'^2 d|t| d\phi d\Omega_\ell}}_{\text{DDVCS}} \xrightarrow{Q'^2 \to 0} \underbrace{\left(\frac{d^4\sigma}{dx_B dQ^2 d|t| d\phi} \right)}_{\text{DVCS}} \frac{\mathcal{N}}{Q'^2} , \tag{2.133}$$

where $\mathcal{N} = \alpha_{\text{EM}}/(3\pi)$ is the remnant of the splitting $\gamma^* \to \mu^-\mu^+$ [80].

Reducing DDVCS to TCS requires:

1. Substituting the electron beam by its own electromagnetic field. In quantum field theory, electrons are surrounded by a sea of virtual photons so by considering the limit where the squared momentum of such photons is zero ($Q^2 \to 0$), the beam can effectively be considered as a source for real photons which scatter off the target. This conception is called the *equivalent photon approximation* (EPA), which was initially developed for classical electromagnetic fields. For its quantum version, cf. [82, 83].

2. Taking away the electron beam, which translates to the TCS reaction

$$\gamma(q) + N(p) \to \gamma^*(q') + N(p')\,, \quad q^2 = 0\,, \tag{2.134}$$

   happening with all particles in the same plane. Consequently, an integration with respect to the azimuthal angle ϕ for the momentum of the final-state proton is required.

3. The small incoming virtuality limit implies the evaluation of CFFs for $\rho \to -\xi$.

Finally, the TCS limit takes the form:

$$\int d\phi \underbrace{\frac{d^7\sigma}{dx_B dQ^2 dQ'^2 d|t| d\phi d\Omega_\ell}}_{\text{DDVCS}} \xrightarrow{Q^2\to 0} \underbrace{\left(\frac{d^4\sigma}{dQ'^2 d|t| d\Omega_\ell}\right)}_{\text{TCS}} \frac{d^2\Gamma}{dx_B dQ^2}\,, \tag{2.135}$$

where Γ is the equivalent photon flux calculable in EPA:

$$\frac{d^2\Gamma}{dx_B dQ^2} = \frac{\alpha_{\text{EM}}}{2\pi Q^2}\left(1 + \frac{(1-y)^2}{y} - \frac{2(1-y)Q^2_{\text{min}}}{yQ^2}\right)\frac{q^0}{Ex_B}\,. \tag{2.136}$$

Here,

$$q^0 = \frac{Q}{\varepsilon} = \frac{Q^2}{2Mx_B} \tag{2.137}$$

is the energy of the photon beam as parameterized in Sect. 2.2, while

$$Q^2_{\text{min}} = \frac{(ym_e)^2}{1-y} \tag{2.138}$$

is the minimum value of the spacelike virtuality evaluated for the electron mass, m_e, which works as a regulator in the calculation of the Γ flux within the EPA formalism. We note that the prescriptions for both DVCS and TCS limits hold for each subprocess, i.e. BH, DDVCS and the interference.

In Fig. 2.6 we show how DDVCS CFFs plotted as a function of ξ evolve as $Q^2 \to 0$ and $Q'^2 \to 0$. The curves for limits, $Q^2 = 0$ and $Q'^2 = 0$, are obtained with independent codes for DVCS and TCS processes available in PARTONS. One can conclude that the limits are reached without any discontinuities, hence DDVCS CFFs exhibit the proper reduction to DVCS and TCS counterparts when one of the two virtualities goes to zero. Although the presented quantity is only the imaginary part of CFF $\mathcal{H}$, results for real parts and other CFFs (not shown here) lead to the same conclusions.

The comparison for cross-sections is shown in Fig. 2.7 for pure VCS subprocesses and in Fig. 2.8 for BH. Also here DVCS and TCS limits are evaluated with independent codes available in PARTONS, which are numerical implementations of works

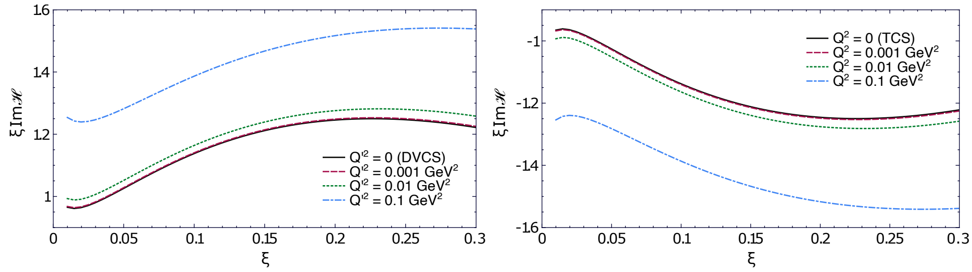

Fig. 2.6. Imaginary part of DDVCS Compton form factor $\mathcal{H}$ as a function of ξ approaching (left) DVCS and (right) TCS limits. The curves for DVCS and TCS limits, corresponding to $Q'^2 = 0$ and $Q^2 = 0$ values, respectively, are obtained with independent codes. Note that these curves nearly overlap with those for $Q'^2 = 0.001$ GeV^2 and $Q^2 = 0.001$ GeV^2. The left (right) plot is for $Q^2 = 1.5$ GeV^2 ($Q'^2 = 1.5$ GeV^2) and $t = -0.15$ GeV^2.

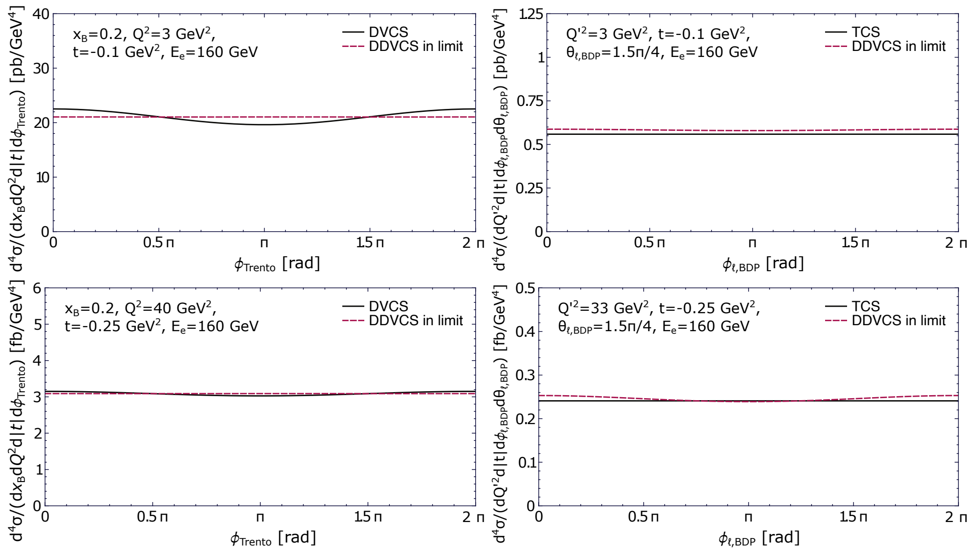

Fig. 2.7. Comparison of DDVCS and (left) DVCS and (right) TCS cross-sections for pure VCS subprocess. Corresponding kinematic configurations are specified in the plots (all are for the fixed target). DDVCS cross-sections are modified according to Eqs. (2.133) and (2.135). Those for DVCS and TCS are evaluated with independent codes.

published in Refs. [84] and [41]. For the GPD, we utilize the Goloskokov-Kroll (GK) model [53, 54] selecting the renormalization and factorization scales to coincide with the energy scale of DDVCS: $\mu_R^2 = \mu_F^2 = Q^2 + Q'^2$.

For pure VCS subprocesses shown in Fig. 2.7 the comparison with the limits is presented for two kinematic configurations, which only differ by either $|t|/Q^2$ or $|t|/Q'^2$ ratios. We see that the relative difference between pure DDVCS and the limits is reduced as these ratios become smaller. This signals that the observed differences stem from kinematic higher-twist corrections, which are related to the choice of the frame used to describe a given process. The impact of these twist corrections is further reduced by the evaluation of the hard part at $t = t_0$, done in order to preserve the electromagnetic gauge invariance, as discussed in Sect. 2.4. The effect is expected, as DVCS and TCS are described in fixed target frames where virtual photons move along the $z-$axis. With two virtual photons in DDVCS case the frame must be different, resulting in differences in the twist expansion. For BH being a pure QED process, we do not deal with this type of expansion and the agreement with the DVCS and TCS limits is exact, as demonstrated in Fig. 2.8.

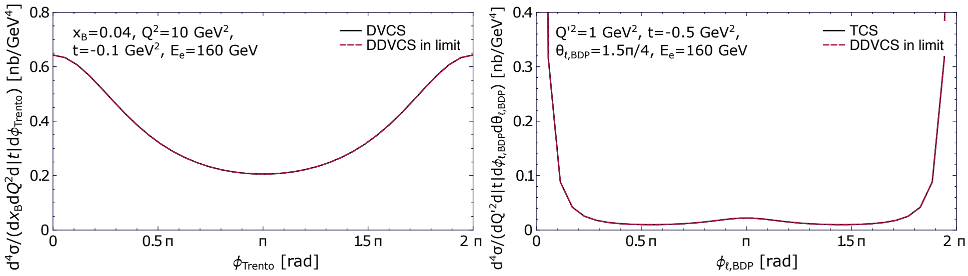

Fig. 2.8. The same as Fig. 2.7 but for BH subprocess.

2.6 Phenomenological estimates

The amplitudes and cross-section of DDVCS and BH we obtained in Sect. 2.4 have been implemented in the open-source PARTONS framework and in the EpIC Monte Carlo (MC) generator, making our work also accessible to experimentalists. This development is important to support the physics case of a new generation of experiments, like JLab12, a possible JLab20+ and the EIC, for which it is desirable to address the feasibility of DDVCS. We will make use of different GPD models: the Goloskokov-Kroll (GK) [53, 54], Vanderhaeghen-Guidal-Guichon (VGG) [55–58] and Mezrag-Moutarde-Sabatié (MMS) [59] models which are already implemented in PARTONS. We note that the MMS model only differs from the GK one by the valence part described by either one-component [85] (MMS) or two-component [25, 86] (GK) double distributions. This difference in the choice of the double distribution makes MMS unique also with respect to VGG, and is responsible for a different behaviour of the model in $x \neq \xi$ domain. This will allow us to address the GPD model dependence. We will do so by checking how DDVCS predictions are affected by the use of different GPD models which give otherwise similar predictions for DVCS and TCS.

In this section we present first results obtained with EpIC for the DDVCS reaction, also checking the accuracy of this generator in reproducing the underlying cross-sections. Our results do not include any simulation of detector effects, and they are not affected by any efficiency one should take into account of in this kind of analysis. Therefore, the presented material should only be considered as a rough estimate and motivation for studying the measurability of DDVCS in more depth.

The distribution of MC events we obtained as a function of y is shown in Fig. 2.9, with the goal of establishing the most favorable region for measuring DDVCS in JLab and EIC facilities. The generation was done for four configurations of electron, E_e, and proton, E_p, beam energies : i) $E_e = 10.6$ GeV and fixed target, ii) $E_e = 22$ GeV and fixed target, iii) $E_e = 5$ GeV and $E_p = 41$ GeV, iv) $E_e = 10$ GeV and $E_p = 100$ GeV; corresponding to JLab12, JLab20+ and EIC experiments. Additional conditions were used in the generation. The range of Q^2 variable was limited to $(0.15, 5)$ GeV^2. The lower value corresponds to the anticipated threshold for detection of scattered electrons. At fixed target, the virtuality of the photon is:

$$Q^2 = 2E_eE_e'(1 - \cos\Theta)\,, \tag{2.139}$$

where E_e' is the energy of the scattered electron and Θ is the angle between this particle and the incoming beam. In experiments such as [87], for values of Q^2 below 0.15 GeV^2, the angle Θ is so small that the scattered electron is lost. To fulfill the exclusivity requirements of DDVCS, the deflected electron must be detected, hence $Q^2 > 0.15$ GeV^2.

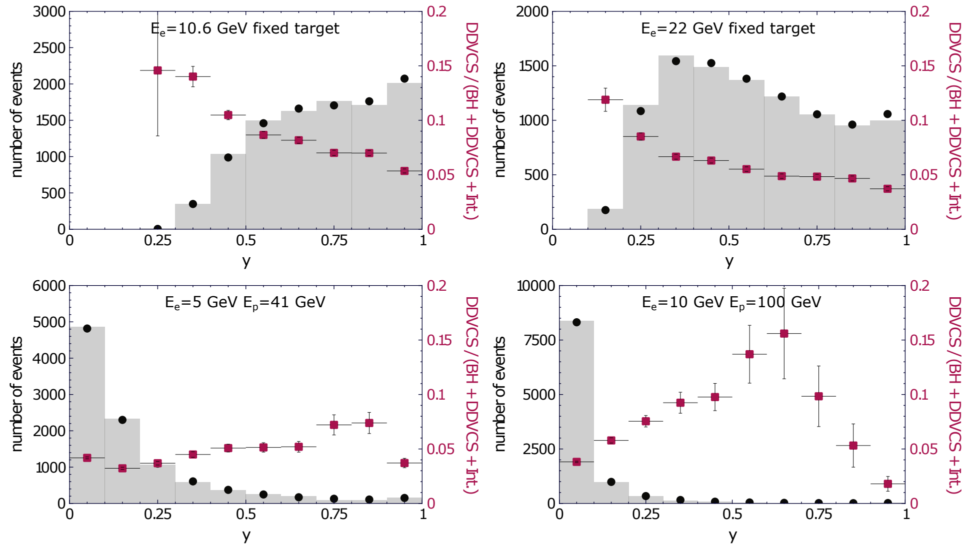

Fig. 2.9. Distributions of Monte Carlo events as a function of the inelasticity variable y. Each distribution is populated by 10000 events generated for the beam energies specified in the plots. Extra kinematic conditions are specified in the text. Black circle markers and gray histograms correspond to the left axes. Reference values for Monte Carlo distributions obtained with a direct integration of differential cross-section. The fraction of events coming from the VCS subprocess with respect to all Monte Carlo events is indicated by red square markers corresponding to the right axes.

The upper cut for Q^2 is a reasonable limit for the observation of cross-section that is suppressed when Q^2 becomes large, vid. Eq. (2.45). The range of Q'^2 is $(2.25, 9)$ GeV^2 corresponding to the region in between meson resonances, already considered in the TCS analysis of Ref. [87]. For $|t|$ we assumed $(0.1, 0.8)$ GeV^2 typical for JLab experiments and $(0.05, 1)$ GeV^2 expected for EIC ones. As for the angular dependencies, the ranges for ϕ and ϕ_ℓ angles are $(0.1, 2\pi - 0.1)$ rad, while for θ_ℓ we have $(\pi/4, 3\pi/4)$ rad. The limitations on angles help to suppress contributions coming from the BH subprocess, which dominate the cross-section. For additional discussion see also [41].

The total cross-section for the scattering (2.2), including all subprocesses, i.e. BH, pure DDVCS and their interference, integrated in the aforementioned kinematic domain is tabulated in Tab. 2.1. In this table we also specify the integrated luminosity needed to record 10000 events presented in Fig. 2.9, and the fraction of events recovered after cutting on the y variable: $(0.1, 1)$ for JLab experiments and $(0.05, 1)$ for EIC ones. This lower cut in y comes from the experimental limitation on determining the energy transferred by the beam, $E_e - E'_e$ (numerator of y (1.21)), in relation with the systematic uncertainty on this quantity due to the detectors precision. We use these graphs to choose the value of y most favorable for observables which is given by a compromise between a large enough fraction of DDVCS events and a non-vanishing cross-section for reaction (2.2). The selected values for the different experiments are gathered in Tab. 2.2, which contains the kinematic points for predictions in Sect. 2.6.1.

In Tab. 2.1 we observe that the cross-section, 4th column, grows for EIC with respect to JLab. The EIC is planned to be an accelerator facility while JLab is a fixed-target experiment, thereby the value of skewness ξ and ρ is expected to be smaller in EIC allowing for accessing a larger sea quark content. As the two facilities have their pluses and minuses, we consider them complementary experiments and provide predictions for both.

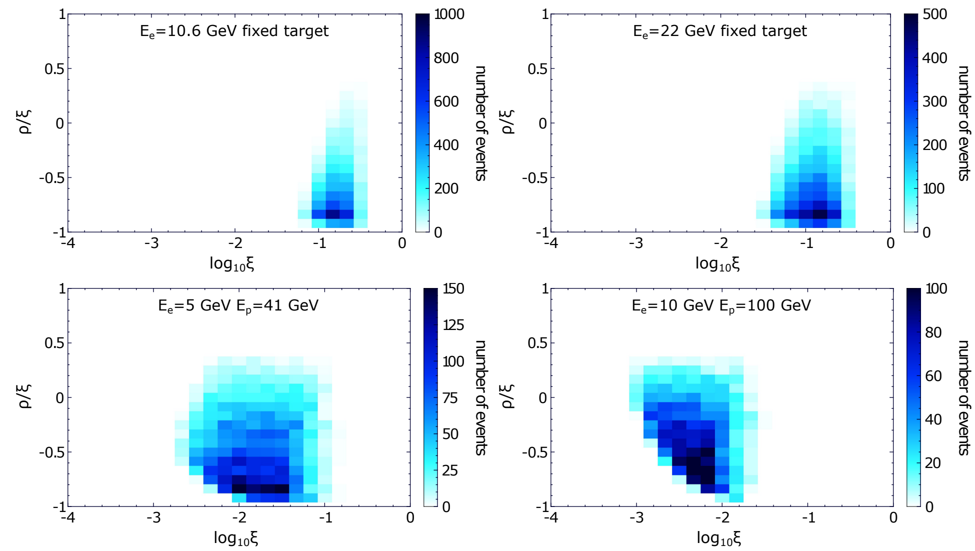

Fig. 2.10. Distribution of Monte Carlo events as a function of the skewness variable ξ and the relative value of generalized Björken variable ρ. Each distribution is populated by 10000 events generated for the DDVCS subprocess at beam energies specified in the plot. Extra kinematical conditions, including cuts on the y variable, are specified in the text.

Experiment	Beam energies [GeV]	Range of $\lvert t\rvert$ [GeV^2]	$\sigma\rvert_{0<y<1}$ [pb]	$\mathcal{L}^{10\mathrm{k}}\rvert_{0<y<1}$ [fb^{-1}]	$y_{\min}$	$\sigma\rvert_{y_{\min}<y<1}/\sigma\rvert_{0<y<1}$
JLab12	$E_e = 10.6$, $E_p = M$	$(0.1, 0.8)$	0.14	70	0.1	1
JLab20+	$E_e = 22$, $E_p = M$	$(0.1, 0.8)$	0.46	22	0.1	1
EIC	$E_e = 5$, $E_p = 41$	$(0.05, 1)$	3.9	2.6	0.05	0.73
EIC	$E_e = 10$, $E_p = 100$	$(0.05, 1)$	4.7	2.1	0.05	0.32

Tab. 2.1. Total DDVCS cross-section including all subprocesses, $\sigma|_{0<y<1}$, obtained for given beam energies under the following conditions: $y \in (0, 1)$, $Q^2 \in (0.15, 5)$ GeV^2, $Q'^2 \in (2.25, 9)$ GeV^2, $\phi, \phi_\ell \in (0.1, 2\pi - 0.1)$, $\theta_\ell \in (\pi/4, 3\pi/4)$ and $|t|$ range specified in 3th column. Corresponding integrated luminosity required to obtain 10000 events is denoted by $\mathcal{L}^{10\mathrm{k}}|_{0<y<1}$. Fraction of events left after restricting the range of y to $(y_{\min}, 1)$ is given in the last column.

In Fig. 2.9 we also show the expected number of events, coming from a direct sevenfold integration with PARTONS of cross-section (2.45) in the aforementioned kinematic domain and limits of y specified by a given bin of the histogram. No free normalization factor is used here: integrated cross-section is multiplied by the luminosity given by EpIC. The comparison between obtained values by PARTONS and MC samples proves the correctness of the generation.

An additional quantity shown in Fig. 2.9 is the fraction of pure DDVCS subprocess in the sample. As expected, this fraction is small, which stresses the need for measuring observables sensitive to the interference between BH and DDVCS. This way observables dependent on the CFFs take advantage of the BH-dominance of the electroproduction of a muon pair.

Additional information is provided in Fig. 2.10, where we show how pure DDVCS events populate the (ξ, ρ) phase-space. Clearly, the $\xi \neq |\rho|$ domain is probed, proving the importance of the DDVCS reaction in the reconstruction of GPDs from experimental data, specially in the *Efremov-Radyushkin-Brodsky-Lepage region* (ERBL)[5] which is not accesisible by DVCS or TCS. We note that because we select a timelike-dominated DDVCS, this is $Q^2 < Q'^2$, we typically access negative values of ρ. DDVCS projects the C-even part of GPDs,

$$H^{(+)}(x, \xi, t) = \sum_f \left(\frac{e_f}{e}\right)^2 \left(H_f(x, \xi, t) - H_f(-x, \xi, t)\right) , \tag{2.140}$$

which can be related to imaginary part of the CFF evaluated at $x = \rho$, vid. Eq. (1.93). Because $H^{(+)}$ is odd in x, the study of DDVCS in this timelike-dominated region (x negative, $x = \rho < 0$) provides a description of DDVCS for the $x > 0$ region too.

Notice that the situation of equal virtualities, $Q^2 = Q'^2$, yields in the LT $\rho = 0$ (2.69). As a result, quark GPDs vanish, $H^{(+)}(\rho = 0, \xi, t) = 0$, signaling that in order to correctly describe this kinematic region one should include gluons or kinematic higher-twist corrections.

2.6.1 DDVCS observables

In this section, we present observables for the kinematics of Tab. 2.2 which are examples of potential measurements of DDVCS at JLab and the EIC, in accordance with the previous discussion. In particular, the value of the inelasticity variable y is obtained from the MC simulations illustrated in Fig. 2.9. From those plots it is also clear that the process is dominated by its pure QED component, the BH subprocess which does no provide access to GPDs. Therefore, we should focus on observables sensitive to the interference between DDVCS and BH. A possible choice is the *single beam-spin asymmetry* (SBSA) defined as follows:

$$A_{LU}(\phi_{\ell,\mathrm{BDP}}) = \frac{\Delta\sigma_{LU}(\phi_{\ell,\mathrm{BDP}})}{\sigma_{UU}(\phi_{\ell,\mathrm{BDP}})} , \tag{2.141}$$

[5] The ERBL region is defined as the kinematic domain for which $|x| < \xi$, while the DGLAP region would correspond to $|x| > \xi$.

where the cross-section difference is taken as

$$\Delta\sigma_{LU}(\phi_{\ell,\text{BDP}}) = \int_0^{2\pi} d\phi \int_{\pi/4}^{3\pi/4} d\theta_{\ell,\text{BDP}} \; \sin\theta_{\ell,\text{BDP}} \times \left(\frac{d^7\sigma^{\rightarrow}}{dx_B dQ^2 dQ'^2 d|t| d\phi d\Omega_{\ell,\text{BDP}}} - \frac{d^7\sigma^{\leftarrow}}{dx_B dQ^2 dQ'^2 d|t| d\phi d\Omega_{\ell,\text{BDP}}} \right) \tag{2.142}$$

and the unpolarized cross-section is

$$\sigma_{UU}(\phi_{\ell,\text{BDP}}) = \int_0^{2\pi} d\phi \int_{\pi/4}^{3\pi/4} d\theta_{\ell,\text{BDP}} \; \sin\theta_{\ell,\text{BDP}} \times \left(\frac{d^7\sigma^{\rightarrow}}{dx_B dQ^2 dQ'^2 d|t| d\phi d\Omega_{\ell,\text{BDP}}} + \frac{d^7\sigma^{\leftarrow}}{dx_B dQ^2 dQ'^2 d|t| d\phi d\Omega_{\ell,\text{BDP}}} \right) . \tag{2.143}$$

Here, we omit the dependence on variables other than angles, while right and left arrows stand for positive and negative helicity of the incoming electron beam, respectively. As indicated before, in order to reduce the contribution coming from the pure BH subprocess, the integration with respect to $\theta_{\ell,\text{BDP}}$ angle is performed in the limited range $(\pi/4, 3\pi/4)$ rad. The cross-section difference $\Delta\sigma_{LU}(\phi_{\ell,\text{BDP}})$ is sensitive to the $\sin\phi_{\ell,\text{BDP}}$ part of the interference, and therefore carries information on the imaginary part of CFFs [42]:

$$\Delta\sigma_{LU}(\phi_{\ell,\text{BDP}}) \propto \text{Im}\left\{ F_1\mathcal{H} - \frac{t}{4M^2}F_2\mathcal{E} + \rho(F_1+F_2)\widetilde{\mathcal{H}} \right\} \times \sin(\phi_{\ell,\text{BDP}}) , \tag{2.144}$$

which is dominated at leading order by the CFF $\mathcal{H}$.

Results in Figs. 2.11 and 2.12 display $\sigma_{UU}(\phi_{\ell,\text{BDP}})$ and $A_{LU}(\phi_{\ell,\text{BDP}})$, respectively, for the kinematics tabulated in Tab. 2.2 and for different GPD models discussed below.

The SBSA, as defined in Eqs. (2.141) and (2.142), reproduces the TCS circular asymmetry [87, 88] in the limit of incoming real photon, $Q^2 \to 0$. An alternative formulation for the SBSA is possible considering an integration with respect to the $\phi_{\ell,\text{BDP}}$ angle. In this case, we get $A_{LU}(\phi)$ by the replacement $\phi_{\ell,\text{BDP}} \leftrightarrow \phi$ in Eq. (2.142). We check that for a timelike-dominated DDVCS ($Q'^2 > Q^2$) and the kinematics of Tab. 2.2, $A_{LU}(\phi)$ is much smaller than $A_{LU}(\phi_{\ell,\text{BDP}})$. Consequently, we do not consider $A_{LU}(\phi)$.

Using Fig. 2.12 we conclude that the magnitude of the asymmetry $A_{LU}(\phi_{\ell,\text{BDP}})$ is up to the order 20% for JLab12, 15% for JLab20+ and 3%-7% for EIC, depending on the GPD model. Such sizable asymmetries and fairly large integrated cross-sections presented in Tab. 2.1 indicates the feasibility of DDVCS programmes at all considered facilities.

In Fig. 2.11, we also depict the cosine components of the cross-section (2.143), defined as

$$\sigma_{UU}^{\cos(n\phi_{\ell,\text{BDP}})}(\phi_{\ell,\text{BDP}}) = M_{UU}^{\cos(n\phi_{\ell,\text{BDP}})} \cos(n\phi_{\ell,\text{BDP}}) , \tag{2.145}$$

through the cosine moments

$$M_{UU}^{\cos(n\phi_{\ell,\text{BDP}})} = \frac{1}{N} \int_0^{2\pi} d\phi_{\ell,\text{BDP}} \; \cos(n\phi_{\ell,\text{BDP}}) \sigma_{UU}(\phi_{\ell,\text{BDP}}) . \tag{2.146}$$

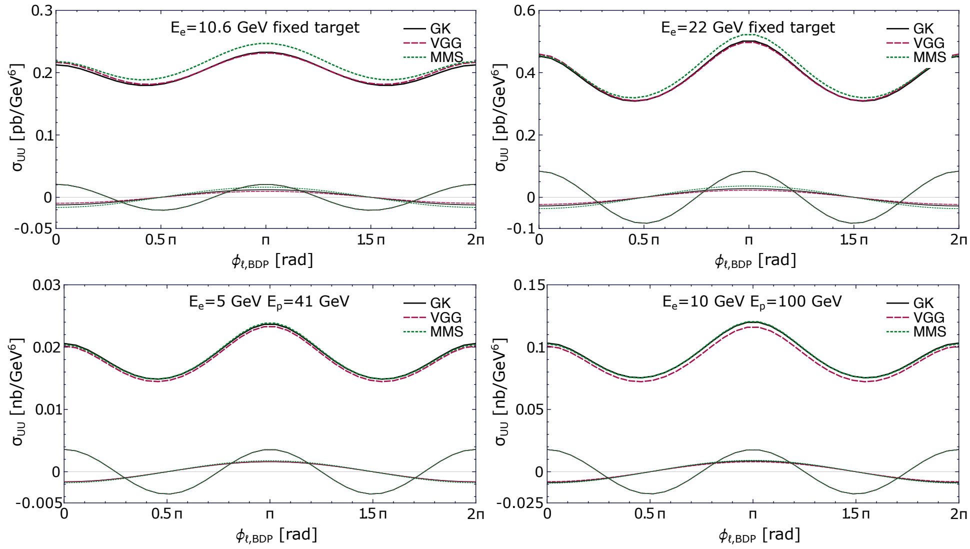

Fig. 2.11. Unpolarized cross-section, $\sigma_{UU}(\phi_{\ell,\mathrm{BDP}})$ in the upper part of each plot, and its $\sigma_{UU}^{\cos\phi_{\ell,\mathrm{BDP}}}(\phi_{\ell,\mathrm{BDP}})$ and $\sigma_{UU}^{\cos 2\phi_{\ell,\mathrm{BDP}}}(\phi_{\ell,\mathrm{BDP}})$ components in the lower part. Calculations made for beam energies specified in the plots and extra kinematic conditions given in Table 2.2. The solid black, dashed red and dotted green curves are for GK, VGG and MMS GPD models, respectively.

Experiment	Beam energies [GeV]	y	$\|t\|$ [GeV^2]	Q^2 [GeV^2]	Q'^2 [GeV^2]
JLab12	$E_e = 10.6$, $E_p = M$	0.5	0.2	0.6	2.5
JLab20+	$E_e = 22$, $E_p = M$	0.3	0.2	0.6	2.5
EIC	$E_e = 5$, $E_p = 41$	0.15	0.1	0.6	2.5
EIC	$E_e = 10$, $E_p = 100$	0.15	0.1	0.6	2.5

Tab. 2.2. DDVCS kinematics used for predictions of asymmetries presented in Figs. 2.11 and 2.12.

Here, $N = 2\pi$ for $n = 0$ and $N = \pi$ for $n > 0$ are the usual normalization factors of a cosine-sine basis for Fourier decomposition. For the interpretation of these contributions we may use Ref. [42]. The constant term, σ_{UU}^{1}, is dominated by the BH subprocess, with a few percent contribution of pure DDVCS, mostly sensitive to the moduli of CFFs. The term $\sigma_{UU}^{\cos\phi_{\ell,\mathrm{BDP}}}$ corresponds to the interference between BH and pure DDVCS, and it carries information about the real parts of CFFs. Finally, $\sigma_{UU}^{\cos 2\phi_{\ell,\mathrm{BDP}}}$ is only sensitive to the BH process. This term vanishes for $\xi = |\rho|$, i.e. is not observed in either DVCS or TCS.

In Figs. 2.11 and 2.12 we make use of three GPD models (GK, VGG and MMS), stressing the usefulness of future DDVCS measurements in regards to the constraining of several types of GPDs. The three models are also depicted in Fig. 2.13 for the dominant distribution probed by DDVCS at leading order,

$$\sum_{q=\{u,d,s\}} e_q^2 H^{q(+)}(x,\xi,t) = H^{(+)}(x,\xi,t)\,, \tag{2.147}$$

where e_q is the fractional electric charge for flavor q relative to proton's and it was denoted: $H^{q(+)}(x,\xi,t) = H^q(x,\xi,t) - H^q(-x,\xi,t)$. We note that all three models are similar in the DGLAP region ($|x| > \xi$), which is a consequence of the common

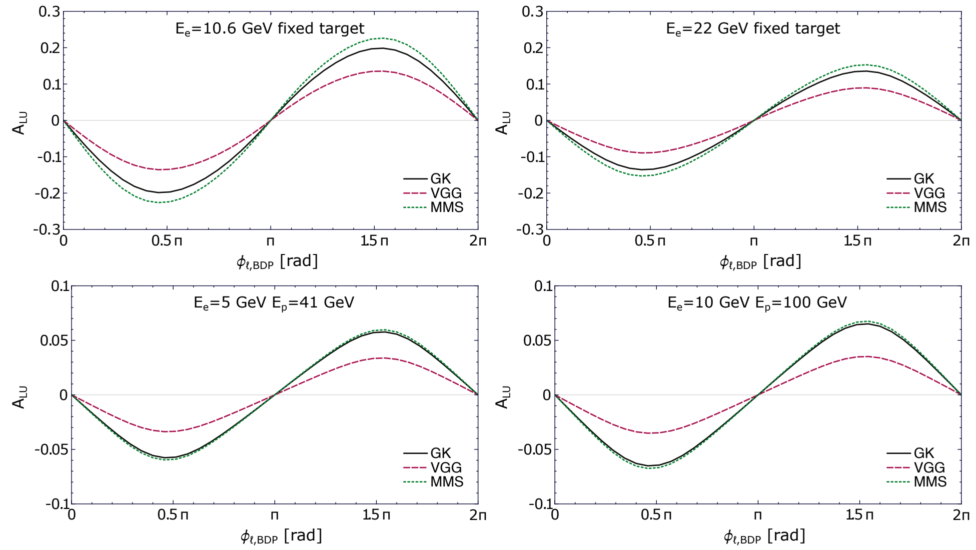

Fig. 2.12. Asymmetry $A_{LU}(\phi_{\ell,\mathrm{BDP}})$ for beam energies specified in the plots and extra kinematic conditions given in Table 2.2. The solid black, dashed red and dotted green curves are for GK, VGG and MMS GPD models, respectively.

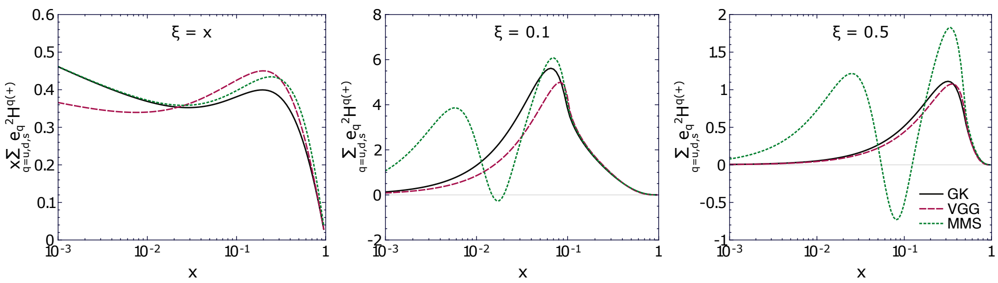

Fig. 2.13. Distributions of $\sum_q e_q^2 H^{q(+)}(x,\xi,t)$ at $t = -0.1\,\mathrm{GeV}^2$, where $q = u,d,s$ flavors for (left) $\xi = x$, (middle) $\xi = 0.1$ and (right) $\xi = 0.5$. The solid black, dashed red and dotted green curves describe the GK, VGG and MMS GPD models, respectively. The C-even part of a given vector GPD is defined as: $H^{q(+)}(x,\xi,t) = H^q(x,\xi,t) - H^q(-x,\xi,t)$. The scale is chosen as $\mu_F^2 = 4\,\mathrm{GeV}^2$.

PDF limit and a similar modelling method, but they differ significantly in the ERBL region ($|x| < \xi$). The latter one is directly probed by DDVCS at LO, making this process a convenient tool to distinguish between such various GPD models.

2.7 Summary and conclusions

In this chapter we considered the theory and phenomenology of the electroproduction of a muon pair (2.2). Its cross-section feeds on two different kind of subprocesses: Bethe-Heitler (BH) which is purely QED and does not provide access to GPDs, and a proper Compton scattering (absorption and subsequent emission of photons by the hadron) known as double deeply virtual Compton scattering (DDVCS). We focus on a spin-1/2 target and particularize the phenomenological predictions for protons at leading order (LO) in the strong coupling constant and kinematic leading twist (LT).

On the theory side, we calculated the Feynman amplitudes for each subprocess by means of the Kleiss-Stirling (KS) spinor techniques, cf. Sect. 2.4.1. The KS methods allow us to obtain a concise set of equations describing the amplitudes in terms of current interactions weighted by the quark content of the hadron. This partonic information is introduced by the Compton form factors (CFFs) that in the case of DDVCS can be used to access GPDs at $x \neq \pm\xi$, as opposed to DVCS and TCS which allow for measurements restricted to $x = \pm\xi$. This approach is complementary to that of a Fourier decomposition by means of the lepton and hadron angles, which can be recovered by making use of the momenta parameterization of Sect. 2.2 in the KS formulation of the amplitudes in Sect. 2.4. Polarization of targets, as well as the DVCS and TCS limits are discussed. By taking the small outgoing/incoming virtuality limits, DDVCS can reproduce the DVCS/TCS results. Differences in the cross-sections stemming from the differences in the reference frames used to described the three processes are explained as kinematic power corrections (kinematic higher twists).

For the impact studies, we implemented the theoretical framework in the open-source `PARTONS` (for numerical estimates) and `EpIC` (for Monte Carlo simulations) softwares. We were able to calculate the unpolarized cross-section (2.143) and other observables such as its cosine components (2.145) and the single beam-spin asymmetry for $\phi_{\ell,\mathrm{BDP}}$ angle (2.141). Our results provide promising results in terms of DDVCS feasibility and encourage further analyses for future experimental facilities.

3

Theoretical framework for the twist decomposition

In this chapter, we explain the notion of twist (Sect. 3.1) and its use as a power counter of light-cone divergences for the product of two generic operators built out of elementary fields. Inclusion of subleading light-cone divergences prompts the kinematic higher-twist corrections after Fourier transform to momentum space. These corrections are effects proportional to powers of $|t|/Q^2$ and M^2/Q^2, where Q represents the energy scale of the considered process.

To the best of our knowledge, Ref. [47] describes the only expansion to all twists in the literature. To achieve such a breakthrough, authors make use of the predictive power of conformal field theories. QCD at LO is one of this kind, although the inclusion of corrections proportional to the strong coupling constant breaks the conformal invariance of QCD at tree level. However, the conformal techniques can still be of use beyond LO by considering the fixed points of the $\beta-$function. The main idea is discussed in Sect. 3.2, which is based on Ref. [89]. The conformal group, its algebra and its application to quantum field theories is considered in Sect. 3.3. There, the main result is the *shadow-operator formalism*, which sets the grounds for the conformal twist expansion. Finally, Sect. 3.4 details the application of the formalism to the Compton tensor (1.60) of two-photon processes such as DVCS, TCS and DDVCS.

This chapter sets the grounds for the calculation presented in chapter 4, where we improve the current theoretical precision of DDVCS off the spin-0 target by including kinematic power corrections. Results for DVCS (which are compared to previous calculations, see Ref. [47]) and TCS are also provided, together with numerical estimates of the kinematic corrections.

3.1 The concept of "twist"

The idea of *twist* is closely related to that of the operator-product expansion (OPE), firstly introduced by K. G. Wilson in 1969 [90]. In the aforementioned publication, Wilson tries to argue methods to write the product of two operators as a series in operators of some basis. Although he did not mention an expansion by means of any quantity under the name of twist, his publication set the grounds for the pioneer work by D. J. Gross and S. B. Treiman [91] where the concept in its *geometric* version was born. Given a generic operator $\mathcal{O}$, its geometric twist τ refers to the quantity:

$$\tau = d_{\mathcal{O}} - j\,, \tag{3.1}$$

where $d_{\mathcal{O}}$ is the energy dimension of the operator $\mathcal{O}$ and j indicates its spin. This way, the geometric twist is related to the irreducible representation of the Lorentz group that the considered operator belongs to. In what follows we will explain the origin of definition (3.1).

As the concept of twist started with Wilson's OPE, let us study a particular example. Consider the free scalar current defined as $j(z) = :\phi(z)\phi(z):$, in a massless theory, where $: \, :$ represents the usual normal ordering. This way we remove the non-zero vacuum expectation value we would get by choosing simply $\phi(z)\phi(z)$ as current. By means of the positive ϕ^+ and negative ϕ^- frequency parts of the field we may rewrite the current as

$$j(z) = \phi^-(z)\phi^-(z) + 2\phi^-(z)\phi^+(z) + \phi^+(z)\phi^+(z)\,. \tag{3.2}$$

Consider now the product

$$\begin{aligned} j(z)j(0) &= :\phi(z)\phi(z): :\phi(0)\phi(0): \\ &= -2\left[\Delta^-(z)\right]^2 - 4i\Delta^-(z):\phi(z)\phi(0): + :\phi(z)\phi(z)\phi(0)\phi(0):\,. \end{aligned} \tag{3.3}$$

The normal-ordered products in the right-hand side of Eq. (3.3) are non-singular as $z^\mu \to 0$, while all singularities are found in complex numbers (not operators) denoted as $\Delta^-(z)$. It represents the following commutator [92]:

$$\begin{aligned} \Delta^-(z) &= i[\phi^+(z), \phi^-(0)] \\ &= -\frac{1}{i}\int \frac{d^4k}{(2\pi)^3} e^{ikz}\theta(-k^0)\delta(k^2-m^2) \\ &= -\frac{i}{4\pi^2}\frac{1}{z^2 - i0z^0}\,. \end{aligned} \tag{3.4}$$

Analiticity at the point $z^\mu = 0$ allows for Taylor expansion around this point:

$$:\phi(z)\phi(0): = j(0) + z_\mu:\partial^\mu\phi(0)\phi(0): + O((z\partial)^2) = j(0) + z_\mu J^\mu(0) + O((z\partial)^2)\,, \tag{3.5}$$

with the definition

$$J^\mu(z) = :\partial^\mu\phi(z)\phi(z):\,. \tag{3.6}$$

With these formulas, Eq. (3.3) reads

$$j(z)j(0) = \frac{1}{8\pi^4(z^2-i0z^0)^2} - \frac{j(0)}{\pi^2(z^2-i0z^0)} - \frac{z_\mu J^\mu(0)}{\pi^2(z^2-i0z^0)} + \frac{O((z\partial)^2)}{z^2-i0z^0} \tag{3.7}$$

$$+ :\phi(z)\phi(z)\phi(0)\phi(0):\,. \tag{3.8}$$

Notice that for short distances ($z^\mu \to 0$), the factor $z_\mu/(z^2 - i0z^0)$ is one power less divergent than $1/(z^2 - i0z^0)$; while for light-cone separations ($z^2 \to 0$) both terms are equally singular. We shall conclude that for short distances

$$j(z)j(0) \xrightarrow[z^\mu \to 0]{} \frac{1}{8\pi^4(z^2 - i0z^0)^2} - \frac{j(0)}{\pi^2(z^2 - i0z^0)} - \frac{z_\mu J^\mu(0)}{\pi^2(z^2 - i0z^0)} - \frac{z_\nu z_\mu \partial^\nu J^\mu(0)}{2\pi^2(z^2 - i0z^0)} + :\phi^4(0): \tag{3.9}$$

and around the light-cone

$$j(z)j(0) \xrightarrow[z^2 \to 0]{} \frac{1}{8\pi^4(z^2 - i0z^0)^2} - \frac{1}{\pi^2(z^2 - i0z^0)} \left[j(0) + z_\mu J^\mu(0) + O(z_\nu z_\mu \partial^\nu \partial^\mu) \right] + :\phi^2(z)\phi^2(0):. \tag{3.10}$$

As Eqs. (3.9) and (3.10) show, for short distances we deal with a finite amount of operators, while for light-cone there is indeed an infinite number of them with equally divergent coefficient functions.

With the knowledge developed so far, we can already propose a formal OPE for the product of any pair of operators built out of fields, namely $A(z)B(0)$, by means of some basis $\{\mathcal{O}^{(n)}\}$. For short distance

$$A(z)B(0) \xrightarrow[z^\mu \to 0]{} \sum_n C_n^{(1)}(z)\mathcal{O}^{(n)}(0), \tag{3.11}$$

with $C_n^{(1)}(z)$ being functions as singular as

$$C_n^{(1)}(z) \sim \left(\frac{1}{z}\right)^{d_A + d_B - d_{\mathcal{O}^{(n)}}}. \tag{3.12}$$

These coefficients can carry indices, coming from Taylor expansion, that have to be contracted with those of $\mathcal{O}^{(n)}(0)$. This kind of indices have been considered implicit.

On the other hand, for light-cone separations

$$A(z)B(0) \xrightarrow[z^2 \to 0]{} \sum_n C_n^{(2)}(z) z^{\mu_1} \cdots z^{\mu_n} \mathcal{O}^{(n)}_{\mu_1 \cdots \mu_n}(0), \tag{3.13}$$

where $C_n^{(2)}(z)$ are divergent functions depending on powers of

$$C_n^{(2)}(z) \sim \left(\frac{1}{z^2}\right)^{\left(d_A + d_B - d_{\mathcal{O}^{(n)}} + n\right)/2}, \tag{3.14}$$

that do not carry indices.

For the case of the time-ordered product, $\mathbb{T}\{A(z)B(0)\}$, the Wilson coefficients carry light-cone divergences produced by inverse powers of $z^2 - i0$, where $i0$ is the Feynman-epsilon prescription. This complex factor relates the upper and lower region of the complex plane for z^0 to the $\theta(z^0)$ and $\theta(-z^0)$ components of the time ordering. Hence, we may write

$$\mathbb{T}\{A(z)B(0)\} \xrightarrow[z^2 \to 0]{} \sum_n C_n^{(3)}(z) z^{\mu_1} \cdots z^{\mu_n} \mathcal{O}^{(n)}_{\mu_1 \cdots \mu_n}(0), \tag{3.15}$$

where

$$C_n^{(3)}(z) \sim \left(\frac{1}{z^2 - i0}\right)^{\left(d_A + d_B - d_{\mathcal{O}^{(n)}} + n\right)/2} . \tag{3.16}$$

The coefficients $C_n^{(1)}$, $C_n^{(2)}$ and $C_n^{(3)}$ are usually referred to as *Wilson coefficients*.

From the expression in Eq. (3.16) we realize that only positive inverse powers would produce a non-zero contribution around the light-cone, i.e.

$$d_A + d_B - d_{\mathcal{O}^{(n)}} + n \geq 0 , \tag{3.17}$$

which leads to

$$d_A + d_B \geq d_{\mathcal{O}^{(n)}} - n . \tag{3.18}$$

Hence, all operators $\mathcal{O}^{(n)}$ satisfying condition (3.18) will contribute near $z^2 \simeq 0$. It is clear that this condition depends not only on the dimension of the operator $d_{\mathcal{O}^{(n)}}$, but also on its order by Taylor expansion n, or in other words on its number of *internal indices*.[1]

Now, consider the Clebsch-Gordan decomposition of a general $n-$ranked Lorentz tensor $\mathcal{O}^{(n)}$ which is given by

$$\mathcal{O}^{(n)}_{\mu_1\mu_2\cdots\mu_n} \sim \left(\frac{n}{2}, \frac{n}{2}\right) \oplus \left(\frac{n-2}{2}, \frac{n-2}{2}\right) \oplus \left(\frac{n-4}{2}, \frac{n-4}{2}\right) \oplus \cdots , \tag{3.19}$$

where the ellipsis contains representations (j_1, j_2) for both $j_1 = j_2 < n/2$ and $j_1 \neq j_2$. Here we use the nomenclature (j_1, j_2) for the representation of spin $j_1 + j_2$ of the Lorentz group. This notation comes from realizing that the algebra of the proper orthocronus Lorentz group, $SO^{\uparrow}(1, 3)$, can be written as the direct sum of two $SU(2)$ algebras. For each of them we assign the spins j_1 and j_2. The irreducible representations (irreps) are obtained when traces over Lorentz indices are removed from each (j_1, j_2) as the operation of taking a trace commutes with the action of the group. In particular, representations with $j_1 = j_2$ are symmetric which are the only ones required as $\mathcal{O}^{(n)}$ is contracted with the symmetric tensor $z^{\mu_1} \cdots z^{\mu_n}$, vid. Eq. (3.15). For further reading on the Lorentz group representations cf. [94].

From Eq. (3.19) we see that the Taylor degree n in Eq. (3.18) coincides with the spin of the highest possible representation of the operator. Consequently, we could change n by the spin of the operator representation, let us denote it as j_n. Then, prior condition (3.18) is re-expressed in the following way: in light-cone OPE, the operators that contribute nearby $z^2 \simeq 0$ are such that their geometric twist defined as $\tau = d_{\mathcal{O}^{(n)}} - j_n$ satisfies to be lesser than the dimension of the operators in the OPE (A and B, $d_A + d_B$):

$$d_A + d_B \geq d_{\mathcal{O}^{(n)}} - j_n = \tau . \tag{3.20}$$

The lowest geometric twist (also known as *geometric leading twist*, LT) corresponds to the operators with the largest spin which, according to the decomposition in Eq. (3.19), belong to the representation $(n/2, n/2)$ for $n-$ranked tensors. Such operators satisfy to be fully symmetric and traceless. Indeed, traces over Lorentz indices are higher-twist operators as they belong to lower spin representations and, as a result, they are accompanied by less divergent Wilson coefficients. This behaviour is

[1] In Eq. (3.13), operators A and B can have their own Lorentz indices that, according to [93], will be referred to as *external*. These indices will be inherited by $\mathcal{O}^{(n)}$ and will label it together with those coming from Taylor expansion. The latter indices are called *internal* and are explicitly written in Eq. (3.13).

expected as traces consist of basically the substitution of two vector-like indices by a metric which after contraction with $z^{\mu_1} \cdots z^{\mu_n}$ in (3.15) yields a positive z^2 power, reducing this way the divergence of the original Wilson coefficient.

Clearly, twist plays no role in short-distance OPE (3.11). In this case, the degree of divergence is specified by $C_n^{(1)}(z)$ and was shown to be of the form

$$C_n^{(1)}(z) \sim \left(\frac{1}{z}\right)^{d_A+d_B-d_{\mathcal{O}^{(n)}}}, \tag{3.21}$$

that will be a singular factor as long as

$$d_A + d_B - d_{\mathcal{O}^{(n)}} \geq 0 \Rightarrow d_A + d_B \geq d_{\mathcal{O}^{(n)}}. \tag{3.22}$$

In other words, the operators that contribute in the short-distance expansion of $A(z)B(0)$ are those whose dimension is smaller than that of $A(z)B(0)$. Neither spin nor twist come into play, i.e. for short-separation OPE the representation of the operator is irrelevant and so is its twist.

Consequently, we have proven that the geometric twist as defined in Eq. (3.1) serves as a counter for the power divergence in the light-cone. This is of special relevance for the processes we are interested in as their cross-section (DIS) and amplitudes (DVCS, TCS, DDVCS) are indeed dominated by distances in the light-cone, issue already discussed in Ch. 1.

Thus, τ is related to operators in position space, not directly to observables which are given in momentum space. As detailed in Ch. 1, the light-cone dominance of the hadron and Compton tensor of inclusive and exclusive processes is a consequence of having a quantity that dominates the kinematics of the scattering and defines the energy scale of the process. Such magnitude is usually the momentum squared of the probe (photons in the case of DIS, DVCS, TCS or DDVCS). When the scale satisfies the conditions of the Björken limit, factorization by means of a convolution between the amplitude of the probe-parton interaction and the distributions of these elemental particles at light-cone distances is an appropriate description of the probe-hadron scattering. Hence, we can conclude:

$$Q^2 \to \infty \Rightarrow z^2 \to 0, \tag{3.23}$$

with Q^2 the aforementioned scale. This connection between the Björken limit and the light-cone suggests the *kinematic* version of the twist notion. By moving away from the light-cone ($z^2 \neq 0$) we effectively relax the Björken conditions, as can be deduced from Eq. (3.23). Since the scale cannot longer be considered infinite, corrections to the light-cone contribution to the hadron and Compton tensors translate to kinematic power corrections of the form

$$\left(\frac{\Lambda^2}{Q^2}\right)^{(\tau_{\text{kin}}-2)/2} < 1, \tag{3.24}$$

where typically Λ^2 is the Madelstam variable t or the hadron mass. Note that the factor 2 which is subtracted from the kinematic twist τ_{kin} is just conventional as, traditionally, the lowest possible twist (the *kinematic leading twist*, LT) has been referred to as *kinematic twist-2*.

Hence, we demonstrated that the two concepts (geometric and kinematic twist) are not disconnected from each other. Nevertheless, there is an important difference between the two notions: the geometric twist is related to the spin of the operator, hence to the irrep of the Lorentz group that the operator belongs to. Conversely, the kinematic twist is related to the matrix elements of the operators in momentum space which relates it to the momenta parameterization, hence to the choice of reference frame. As a result, the geometric twist is a Lorentz invariant, whereas the kinematic twist is not.

Although the explicit decomposition of the Compton tensor to all kinematic twists will be detailed in Sect. 3.4 and practical calculations for DDVCS will be developed in Ch. 4, to this point we can anticipate the result and sketch how these two types of twist are related regarding the Compton tensor. The techniques we will employ are those of V. M. Braun, Y. Ji and A. N. Manashov [46, 47] which are grounded on *conformal symmetry* and *conformal field theory* (CFT). Accordingly, we can sketch the kinematic-twist decomposition of the Compton tensor as follows (see explanation below):

$$
\begin{aligned}
T^{\mu\nu}_{s_2 s_1} &= i\int d^4z\, e^{iq'z}\langle p', s_2|\mathcal{T}\left\{j^\nu(z)j^\mu(0)\right\}|p, s_1\rangle \\
&\sim i\int d^4z\, e^{iq'z}\frac{f^{\mu\nu}(z,\partial)}{(-z^2+i0)^J}\underbrace{\left[e^{-i\ell z}\right]_{\mathrm{LT}}}_{\textbf{geom. LT}} \longrightarrow \textbf{geom. LT} \text{ to connect to the usual } \textbf{GPDs} \\
&\sim i\int d^4z\int_0^1 dw\, e^{i(q'-w\ell)z}\frac{\widetilde{f}^{\mu\nu}(z,w)}{(-z^2+i0)^K} \\
&\sim \sum_{n,m} f^{\mu\nu}_{n,m}(\Delta,\bar{p},q')\times I_{n,m} \qquad \longrightarrow \qquad I_{n,m} = \int_0^1 dw\,\frac{w^n}{(\ell^2w^2-2q'\ell w+Q'^2+i0)^m} \\
&\sim \underbrace{O(1)}_{\textbf{kin. LT}} + \left(\text{powers of } \frac{|t|}{-2q'\Delta}, \frac{M^2}{-2q'\Delta}\right).
\end{aligned}
\tag{3.25}
$$

In second line of Eq. (3.25), it was introduced the general form of the matrix elements of the product of currents (index J is a positive integer, likewise for K in third line), where we specified the geometric LT projection of the exponential dependent on the hadron momenta through the vector $\ell = \ell(\bar{p}, \Delta)$. This projection can be found in Eq. (D.19) of App. D and selects the basis of operators $\mathcal{O}^{(n)}$ of the OPE (3.15) to be that of the usual GPDs. This way the calculation will consist of a series of hard-coefficient functions convoluted with GPDs and their derivatives. This projection is a series of integrals in the auxiliary variable w of the third line, which after Fourier transform yields an expression of the Compton tensor as a superposition of the different integrals $I_{n,m}$. These ones are explicitly formulated on the right hand side of the fourth line. Ultimately, as suggested by the momentum dependence of the integrals $I_{n,m}$, we can read out the scale for a general two-photon process (e.g. DDVCS, DVCS and TCS) to be:

$$
\mathrm{Q}^2 = -2q'\Delta = Q^2 + Q'^2 + t\,, \tag{3.26}
$$

due to $-2q'\ell \propto -2q'\Delta$. Thereby, we can expand in powers of

$$
\frac{|\ell^2|}{-2q'\Delta} < 1\,, \tag{3.27}
$$

which, taking into account that ℓ is a combination of $\bar{p} = (p + p')/2$ and $\Delta = p' - p$, is equivalent to a series in powers of

$$\frac{|t|}{-2q'\Delta} < 1 \quad \text{and} \quad \frac{M^2}{-2q'\Delta} < 1 \,. \tag{3.28}$$

This expansion is sketched in the last line of Eq. (3.25), denoting as kinematic LT the term that survives in the $\mathbb{Q}^2 \to \infty$ limit. In $\mathbb{Q}^2$ (3.26) we can keep the factor t order-by-order in the kinematic-twist expansion as the difference is of the next twist.

This chapter is organized as follows: in Sect. 3.2 we will discuss the validity of the conformal symmetry in QCD as the employed techniques are based on CFT. In Sect. 3.3, we will study the features of CFTs that will be used later on in Sect. 3.4 to describe the modern techniques for the kinematic-twist decomposition developed by V. M. Braun, Y. Ji and A. N. Manashov in the last decade [46, 47].

As a final remark, in Ch. 4, we will make use of the formalism described in Sect. 3.4 to calculate the different twist components of the DDVCS amplitude and, by taking the appropriate limits, of DVCS and TCS.

3.2 Validity of conformal symmetry in QCD

The main object in a quantum field theory (QFT) is the correlator of fields. Endowing a QFT with conformal symmetry makes it possible to link correlators of three operators to the Wilson coefficients of the OPE, as it will be shown later. A conformal field theory describes interactions that are invariant under the *conformal group*, which is nothing but the set of transformations that modifies the spacetime metric by a local or global multiplicative factor. It is, therefore, an extension of the Poincaré group which consists of Lorentz transformations plus translations. For being an extension, the number of symmetries that the correlator must suffice is larger than that of the Poincaré group alone, allowing us to fully determine the structure of the correlators up to some constant. This constant is to be fixed by physical constrains. This predictive power is what makes CFT a useful framework to work out the OPE.

In the case of massless QCD, the Lagrangian is invariant under conformal transformations so it is classically a CFT. Nevertheless, quantum corrections, which are dependent on the $\beta-$function, break the symmetry. For a further discussion on this topic cf. [95]. This issue begs the following question: why should we be interested in this kind of transformations if, as a matter of fact, QCD is not conformally invariant? The answer lays on the renormalization group flow.

In QFT we describe dynamics via the Lagrangian of the theory which contains all degrees of freedom (particles) and the couplings among them. When considering processes below a certain energy scale, μ_R, we can perform an *integration-out* of the degrees of freedom which corresponds to a formal removal of the energy modes higher than the chosen scale μ_R. This procedure yields an *effective* Lagrangian, different from the original one, where the effect of the higher energy modes is hidden in a new set of couplings that now depend on μ_R. This dependence, or *running*, of the coupling parameter ($\mathrm{g_s}$) is described via the $\beta-$function:

$$\beta(\mathrm{g_s}) = \frac{\partial \mathrm{g_s}}{\partial \ln \mu_\mathrm{R}} = \mu_\mathrm{R} \frac{\partial \mathrm{g_s}}{\partial \mu_\mathrm{R}} \,.$$

The modification in g_s with rate β can be interpreted as a transition in the space of all possible Lagrangians. This is usually referred to as the *renormalization group flow* (RG flow). The topology of this space of Lagrangians can be understood via the *fixed points* of the theory. A fixed point is given by a particular value of the coupling, g_{s*}, for which $\beta(g_{s*}) = 0$. In this case, the coupling constant is an actual constant, with no dependence on the scale μ_R, hence scale invariant.

But how do these fixed points control the RG flow? Let us consider the space of Lagrangian parameters in the neighbourhood of the fixed point g_{s*}. This point is considered *stable* if trajectories in said space flow towards g_{s*}, whereas it is *unstable* otherwise. There are also *marginal* directions along which the coupling constant does not change. As a consequence, we can consider our non-conformally invariant QCD as the result of some CFT that has been perturbed away from a particular fixed point. For this purpose, one may consider a QCD in spacetime dimension $D = 4 - 2\epsilon$, $\epsilon > 0$, for which the $\beta-$function takes the form [89, 96]

$$\beta(a) = -2a\left(\epsilon + \beta_0 a + \beta_1 a^2 + O\left(a^3\right)\right) , \tag{3.29}$$

where

$$a = \frac{\alpha_s}{4\pi}, \quad \alpha_s = \frac{g_s^2}{4\pi}, \quad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f, \quad \beta_1 = \frac{2}{3}\left(17N_c^2 - 5N_cN_f - \frac{3}{2}\frac{N_c^2-1}{N_c}N_f\right). \tag{3.30}$$

Here, N_c is the number of colors and N_f is that of flavors. For $N_f > 11N_c/2$, we have $\beta_0 < 0$. This implies the existence of a fixed point, $a_*(\epsilon)$, that can be expressed as a series in powers of ϵ from the condition $\beta(a_*) = 0$:

$$a_*(\epsilon) = -\frac{\epsilon}{\beta_0} - \left(\frac{\epsilon}{\beta_0}\right)^2\frac{\beta_1}{\beta_0} + O(\epsilon^3). \tag{3.31}$$

This is the so-called *Fisher-Wilson fixed point* [97]. In order to remove the divergences of the theory renormalization is required, delivering renormalized operators of the form

$$[\mathcal{O}_k]_R = \sum_j \mathbb{Z}_{kj}\mathcal{O}_j , \tag{3.32}$$

where the sum includes all operators that mix under evolution and, therefore, have the same quantum numbers. The renormalization matrix $\mathbb{Z}$ is calculable considering Green functions as detailed in [98] and it can be used to define the anomalous dimension matrix:

$$\mathbb{H} = \mathbb{Z}^{-1}(\mu_R\partial_{\mu_R}\mathbb{Z}). \tag{3.33}$$

With $\mathbb{H}$ the RG equations for QCD at $D = 4 - 2\epsilon$, at the fixed point a_*, take the form

$$\left(\mu_R\partial_{\mu_R} + \mathbb{H}(a_*)\right)[\mathcal{O}]_R = 0. \tag{3.34}$$

In renormalization based on minimal-subtraction (MS) schemes, $\mathbb{H}$ does not depend on the parameter ϵ by design. Therefore, it is the same for both $a_*(\epsilon)$ and $a(\epsilon = 0)$, the latter being the coupling constant for QCD at the spacetime dimension $D = 4$ for which RG equations are

$$\left(\mu_R\partial_{\mu_R} + \beta(a)\partial_a + \mathbb{H}(a)\right)[\mathcal{O}]_R = 0. \tag{3.35}$$

As a result, one can work with QCD on $D = 4 - 2\epsilon$ at the fixed point (3.31), which is an exact conformal field theory, and return to the QCD on $D = 4$ with generic coupling a by simply writing ϵ in terms of a_*, and then substituting $a_* \to a$. As it was shown in Ref. [89], this methodology holds for an arbitrary number of flavors, hence the requirement of large N_f which was considered previously for the existence of the fixed point is not a restriction for the use of conformal symmetry.

This discussion supports the use of the conformal group in the context of QCD, even beyond LO. At LO, $a = 0$ so that the $\beta-$function vanishes and we can consider QCD on $D = 4$ to be an exact conformal field theory. In this case, anomalous dimensions of operators are zero.

In the next sections, we will study the conformal group as well as the tools to implement it in field theories, in particular to predict the structure of correlators and the OPE for different kinds of fields. This will lead us to the so-called *shadow-operator formalism* setting the grounds for the conformal OPE developed in Ref. [46].

3.3 Conformal group and its algebra

Given a general metric at a spacetime point z, $g_{\mu\nu}(z)$, conformal transformations are defined as the change of coordinates $z \to z'$ that renders a re-scaling of the metric

$$g_{\mu\nu}(z) \to g'_{\mu\nu}(z') = \Omega^2(z) g_{\mu\nu}(z)\,, \tag{3.36}$$

with $\Omega(z)$ being some function. For $\Omega(z) \equiv 1$, the conformal group reduces to Poincaré's. As a consequence, conformal transformations preserve the angle between two vectors $u\,, v$,

$$\theta = \frac{uv}{\sqrt{u^2 v^2}}\,. \tag{3.37}$$

Clearly, these transformations do not modify the light-cone distances.

Let us consider infinitesimal coordinate transformations, $z^\mu \to z'^\mu = z^\mu + \epsilon^\mu(z)$ with $\epsilon^\mu(z)/z^\mu \ll 1\,\forall\,\mu$. Under such transformations, the metric gets modified in the following way:

$$g'_{\mu\nu}(z') = \frac{\partial z^\rho}{\partial z'^\mu}\frac{\partial z^\sigma}{\partial z'^\nu} g_{\rho\sigma}(z) = g_{\mu\nu}(z) - \left(\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu\right)\,. \tag{3.38}$$

We keep only first order in $\epsilon(z)$ and use the notation $\partial_\mu = \partial/\partial z^\mu$.

According to transformation (3.36), for the previous result to represent a conformal transformation we must require

$$\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = f(z) g_{\mu\nu}(z)\,, \tag{3.39}$$

for some function $f(z)$. Considering Minkowsky spacetime

$$g_{\mu\nu}(z) \equiv g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)\,, \tag{3.40}$$

and contracting indices in Eq. (3.39) with the metric, you can prove that

$$f(z) = \frac{2}{D}\partial\epsilon(z) \Rightarrow \Omega^2(z) = 1 - \frac{2}{D}\partial\epsilon(z)\,, \tag{3.41}$$

to first order in $\epsilon(z)$.

To find a concise equation for ϵ, let us act on Eq. (3.39) with ∂^ν obtaining [99–104]

$$\partial_\mu(\partial\epsilon) + \partial^2\epsilon_\mu = \frac{2}{D}\partial_\mu(\partial\epsilon)\,. \tag{3.42}$$

Now, acting on this last expression with ∂_ν, exchanging indices μ,ν in the result, adding it back and using again Eq. (3.39),

$$(g_{\mu\nu}\partial^2 + (D-2)\partial_\mu\partial_\nu)\partial\epsilon = 0\,. \tag{3.43}$$

Contracting indices with the metric again,

$$(D-1)\partial^2(\partial\epsilon) = 0\,. \tag{3.44}$$

Eq. (3.44) holds for $D \neq 1$ only. In particular, the case $D = 2$ that turns out to be richer than other dimensions regarding conformal group features, for a reference see [99, 100].

Equation (3.44) can be fully solved. First, realize that $\epsilon(z)$ can be at most of second order in z, so we could propose the following *Ansatz*:

$$\epsilon_\mu(z) = a_\mu + b_{\mu\nu}z^\nu + c_{\mu\nu\rho}z^\nu z^\rho\,. \tag{3.45}$$

Because the group elements are independent of the particular z, we can study each term separately. The constant term a_μ plays the role of an infinitesimal translation and the generator, P_μ, has the usual representation

$$P^\nu = i\partial^\nu\,. \tag{3.46}$$

The linear term in (3.45) can be introduced in Eq. (3.44), so that the corresponding transformation on the spacetime coordinate takes the form:

$$z_\mu \to z_\mu + b_{\mu\nu}z^\nu = \left(1 + \frac{1}{D}b^\sigma_\sigma\right) z_\mu + b_{[\mu\nu]}z^\nu\,. \tag{3.47}$$

The first term in the RHS are scale transformations or dilations, this is $z_\mu \to (1+\alpha)z_\mu$ identifying $\alpha = b^\sigma_\sigma/D$. Taking into account that the Lorentz group is a subgroup of the conformal one, the antisymmetric part must correspond to an infinitesimal Lorentz transformation:[2]

$$b_{[\mu\nu]}z^\nu = \widetilde{b}_{\rho\nu}(z^\nu\partial^\rho - z^\rho\partial^\nu)z_\mu\,, \tag{3.48}$$

where we defined $\widetilde{b}_{\rho\nu} = b_{\rho\nu}/2$.

We can reproduce the transformation in Eq. (3.47) if we take the representation of the dilation operator to be

$$\mathcal{D} = iz\partial\,, \tag{3.49}$$

while for the Lorentz generators:

$$L_{\mu\nu} = i(z_\mu\partial_\nu - z_\nu\partial_\mu)\,. \tag{3.50}$$

Finally, we are left with the quadratic term in (3.45). For this one, Eq. (3.44) does not

[2] In literature of the conformal group, Lorentz transformations are usually referred to as *rotations*.

provide conditions so we need to go back to Eq. (3.39). Acting on it with ∂_ρ, permuting the indices and suitably combining the three resultant differential equations, we arrive to

$$2\partial_\mu\partial_\nu\epsilon_\rho = \frac{2}{D}(g_{\mu\nu}\partial_\rho - g_{\rho\mu}\partial_\nu - g_{\nu\rho}\partial_\mu)(\partial\epsilon)\,. \tag{3.51}$$

Introducing the solution (3.45), the above equation renders the following expression for $c_{\mu\nu\rho}$:

$$c_{\mu\nu\rho} = -g_{\mu\rho}c_\nu - g_{\mu\nu}c_\rho + g_{\nu\rho}c_\mu\,, \quad c_\mu = \frac{c^\nu{}_{\nu\mu}}{D}\,, \tag{3.52}$$

which translates into the so-called *special conformal transformations*

$$\begin{aligned} z_\mu \to z_\mu - 2(zc)z_\mu + z^2c_\mu + \underbrace{O\left((cz)^2\right)}_{cz\ll 1} &= (1 - 2(zc)(z\partial) + z^2(c\partial))z_\mu + O\left((cz)^2\right) \\ &= \exp\{-ic_\nu K^\nu\}z_\mu\,. \end{aligned} \tag{3.53}$$

So we can read out the following representation of the generator for the special conformal transformations:

$$K^\nu = -i(2z^\nu(z\partial) - z^2\partial^\nu)\,. \tag{3.54}$$

Gathering all operators that account for the transformation (3.45) we get the following representations for the generators of the conformal group in $\mathbb{R}^D$ space:

$$\begin{cases} P_\mu = i\partial_\mu\,, \\ \mathcal{D} = iz\partial\,, \\ L_{\mu\nu} = i(z_\mu\partial_\nu - z_\nu\partial_\mu)\,, \\ K_\mu = -i(2z_\mu(z\partial) - z^2\partial_\mu)\,. \end{cases} \tag{3.55}$$

which define the conformal algebra:

$$\begin{cases} [\mathcal{D}, P^\nu] = -iP^\nu\,, \\ [\mathcal{D}, K^\nu] = iK^\nu\,, \\ [K^\mu, P^\nu] = 2i(L^{\mu\nu} + g^{\mu\nu}\mathcal{D})\,, \\ [K^\rho, L^{\mu\nu}] = i(g^{\rho\mu}K^\nu - (\mu \leftrightarrow \nu))\,, \\ [P^\rho, L^{\mu\nu}] = i(g^{\rho\mu}P^\nu - (\mu \leftrightarrow \nu))\,, \\ [L^{\mu\nu}, L^{\rho\sigma}] = i(g^{\nu\rho}L^{\mu\sigma} + g^{\mu\sigma}L^{\nu\rho} - g^{\mu\rho}L^{\nu\sigma} - g^{\nu\sigma}L^{\mu\rho})\,. \end{cases} \tag{3.56}$$

All other commutators vanish.

3.3.1 Jacobian matrix

After providing the coordinates transformation and having identified the group generators and their algebra, we are still missing an important element: the Jacobian of the change of coordinates. A conformal transformation affects the path integral of the corresponding QFT, and hence the correlators, by modifying the fields (which we will detail in Sect. 3.3.3) and the coordinates. Therefore, the Jacobian of the change of coordinates is required as it will enter the calculation of the conformally transformed QFT action.

Using Eqs. (3.39) and (3.41), and keeping only $O(\epsilon)$ terms

$$\frac{\partial z^\mu}{\partial z'^\nu} = \delta^\mu_\nu - \partial_\nu \epsilon^\mu = \delta^\mu_\nu \left(1 - \frac{2}{D}\partial\epsilon\right) + \partial^\mu \epsilon_\nu = \delta^\mu_\nu \left(1 - \frac{1}{D}\partial\epsilon\right) - \underbrace{\frac{\delta^\mu_\nu}{D}\partial\epsilon}_{(\partial^\mu\epsilon_\nu + \partial_\nu\epsilon^\mu)/2} + \partial^\mu\epsilon_\nu$$
$$= \left(1 - \frac{1}{D}\partial\epsilon\right)\left(\delta^\mu_\nu + \frac{1}{2}(\partial^\mu\epsilon_\nu - \partial_\nu\epsilon^\mu)\right) . \quad (3.57)$$

Let us show that the first part of this Jacobian matrix corresponds to infinitesimal dilations, while the second to Lorentz transformations.

For dilations, $z^\mu \to z'^\mu = (1+\alpha)z^\mu$ so $\epsilon^\mu = \alpha z^\mu$, being α a constant. Then,

$$\frac{\partial z^\mu}{\partial z'^\nu} = (1-\alpha)\delta^\mu_\nu \equiv \left(1 - \frac{1}{D}\partial\epsilon\right)\delta^\mu_\nu , \quad (3.58)$$

which is the first term of Eq. (3.57).

Because the second term in Eq. (3.57) is antisymmetric, only Lorentz transformations can contribute to it. Indeed, under the Lorentz group, $z^\mu \to z'^\mu = R^\mu{}_\nu z^\nu = (\delta^\mu_\nu + \omega^\mu{}_\nu)z^\nu$ so $\epsilon^\mu = \omega^\mu{}_\nu z^\nu$, with $\omega^\mu{}_\nu = -\omega_\nu{}^\mu$. Therefore,

$$\partial^\mu\epsilon_\nu - \partial_\nu\epsilon^\mu = -\omega^\mu{}_\alpha\delta^\alpha_\nu + \partial^\mu(\omega_\nu{}^\alpha z_\alpha) = -\omega^\mu{}_\nu + \omega_\nu{}^\mu = -2\omega^\mu{}_\nu . \quad (3.59)$$

The Jacobian for Lorentz transformations is

$$\frac{\partial z^\mu}{\partial z'^\nu} = \delta^\mu_\nu - \omega^\mu{}_\nu \equiv \delta^\mu_\nu + \frac{1}{2}(\partial^\mu\epsilon_\nu - \partial_\nu\epsilon^\mu) , \quad (3.60)$$

which correspond to the second term in Eq. (3.57).

With these results in mind, we can conclude that for finite conformal transformations it is true that

$$\frac{\partial z^\mu}{\partial z'^\nu} = \Omega(z) R_\nu{}^\mu \Rightarrow dz^\mu = \Omega(z) R_\nu{}^\mu dz'^\nu , \quad (3.61)$$

where $\Omega(z)$ is the function that controls the re-scaling of the metric under conformal transformations, vid. Eq. (3.36), and R is a Lorentz transformation.

3.3.2 Inversion tensor

Special conformal transformations are complicated to deal with, so the common practice is to make use of the related transformations called *inversions*. They are defined as

$$z^\mu \to \mathcal{I}z^\mu = \frac{z^\mu}{z^2} \equiv \left(\frac{1}{z}\right)^\mu , \qquad \mathcal{I}^2 = 1 , \quad (3.62)$$

in which case

$$\frac{\partial(\mathcal{I}z^\mu)}{\partial z^\nu} = \frac{1}{z^2}\mathcal{I}^\mu_\nu(z) \Rightarrow d(\mathcal{I}z^\mu) = \frac{1}{z^2}\mathcal{I}^\mu_\nu(z) dz^\nu . \quad (3.63)$$

Here we defined the inversion tensor as

$$\mathcal{I}^\mu_\nu(z) = \delta^\mu_\nu - 2\frac{z^\mu z_\nu}{z^2} . \quad (3.64)$$

By definition, inversions are discrete transformations not connected to the identity, $\det(\mathcal{I}^{\mu}_{\ \nu}) = -1$, and therefore cannot be represented as an exponential of some generator. Nevertheless, they can be used to generate special conformal transformations thanks to the following relation:

$$K^{\mu} = \mathcal{I}P^{\mu}\mathcal{I} \Rightarrow \exp\{-icK\} = \mathcal{I}\exp\{-icP\}\mathcal{I}\,, \quad c \in \mathbb{R}^D\,, \tag{3.65}$$

and, consequently, the finite special conformal transformation is

$$z^{\mu} \xrightarrow{\mathcal{I}} \frac{z^{\mu}}{z^2} \xrightarrow{e^{-icP}} \frac{z^{\mu}}{z^2} + c^{\mu} \xrightarrow{\mathcal{I}} \frac{\frac{z^{\mu}}{z^2} + c^{\mu}}{\left(\frac{z}{z^2} + c\right)^2} = \frac{z^{\mu} + c^{\mu}z^2}{1 + 2cz + c^2z^2}\,. \tag{3.66}$$

Relation (3.65) allows to substitute special conformal transformations by inversions when studying covariance and invariance under the conformal group. Let us write down some useful properties of the inversion tensor:

$$\mathcal{I}_{\mu\nu}(z) = \mathcal{I}_{\mu\nu}\left(\frac{1}{z}\right)\,, \tag{3.67}$$

$$\mathcal{I}_{\mu\sigma}(z)\mathcal{I}^{\sigma}_{\nu}(z) = g_{\mu\nu}\,, \tag{3.68}$$

$$\mathcal{I}_{\mu\nu}(z)z^{\nu} = -z_{\mu}\,, \tag{3.69}$$

$$\mathcal{I}_{\mu\nu}(z-y) = \mathcal{I}^{\rho}_{\mu}(z)\mathcal{I}^{\sigma}_{\nu}(y)\mathcal{I}_{\rho\sigma}\left(\frac{1}{z} - \frac{1}{y}\right)\,. \tag{3.70}$$

Also, denoting conformal transformed coordinates as z', y', it holds:

$$\mathcal{I}_{\mu\nu}(z-y) = \mathcal{I}_{\rho\sigma}(z'-y')R^{\rho}_{\ \mu}R^{\sigma}_{\ \nu}\,. \tag{3.71}$$

3.3.3 Conformal action on fields

In QFTs, symmetries are realized through operators built out of fields and acting on the Hilbert space. A common practice is to consider these operators in Heisenberg image and, consequently, as dependent on spacetime coordinates. A general field $\Phi(z)$ in this image can be written as

$$\Phi(z) = e^{izP}\Phi(0)e^{-izP}\,, \tag{3.72}$$

and by taking its derivative we find

$$\partial_{\mu}\Phi(z) = ie^{izP}\left(P_{\mu}\Phi(0) - \Phi(0)P_{\mu}\right)e^{-izP} = i[P_{\mu}, \Phi(z)] \Rightarrow [P_{\mu}, \Phi(z)] = -i\partial_{\mu}\Phi(z)\,. \tag{3.73}$$

In App. E there is a discussion about the relation between the action of the group on fields, the (representation of) generators and the commutators of these two objects. According to the reasoning there and denoting by δ_G the transformation induced by generator G at a point z,

$$\delta_G\Phi(z) = \Phi'(z) - \Phi(z)\,, \tag{3.74}$$

we can conclude that translations acting on coordinates as $z^{\mu} \to z^{\mu} + a^{\mu}$ have an action on fields given by

$$\delta_P\Phi(z) = -a^{\mu}\partial_{\mu}\Phi(z)\,. \tag{3.75}$$

From commutator (3.73) the representation[3] of the generator of translations is $P^\mu = i\partial^\mu$.

In order to find the action of the remaining generators on local operators we start with the *stability group*, which is the subgroup that leaves the origin $z = 0$ invariant. The commutators of these generators with operators $\Phi(0)$ are

$$\begin{aligned} [\mathcal{D}, \Phi(0)] &= -i\hat{\boldsymbol{\ell}}\Phi(0)\,, \\ [L_{\mu\nu}, \Phi(0)] &= iS_{\mu\nu}\Phi(0)\,, \\ [K_\mu, \Phi(0)] &= 0\,, \end{aligned} \tag{3.76}$$

where S is the spin matrix carrying both Lorentz and spinorial indices, and $\hat{\boldsymbol{\ell}}$ an operator such that $\hat{\boldsymbol{\ell}}\Phi(0) = \ell\Phi(0)$ for a field Φ with scaling dimension $\ell \in \mathbb{R}$. The spin matrix acts on scalar $\Phi = \phi$, fermion $\Phi = \psi$ and vector $\Phi = A^\rho$ fields as follows

$$S_{\mu\nu}\phi(z) = 0\,, \quad S_{\mu\nu}\psi(z) = \frac{i}{2}\sigma_{\mu\nu}\psi(z)\,, \quad S_{\mu\nu}A^\rho(z) = g_\nu^{\ \rho}A_\mu(z) - (\mu \leftrightarrow \nu)\,, \tag{3.77}$$

with

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right] \,. \tag{3.78}$$

Using Eqs. (3.72), (3.76) and the Hausdorff formula,[4] the commutators of generators and fields at a given point z may be written as

$$[\mathcal{D}, \Phi(z)] = -i(\hat{\boldsymbol{\ell}} + z\partial)\Phi(z)\,, \tag{3.79}$$

$$[L_{\mu\nu}, \Phi(z)] = -i(z_\mu\partial_\nu - z_\nu\partial_\mu - S_{\mu\nu})\Phi(z)\,, \tag{3.80}$$

$$[K_\mu, \Phi(z)] = i(2z_\mu(z\partial) - z^2\partial_\mu + 2\hat{\boldsymbol{\ell}}z_\mu - 2z^\nu S_{\mu\nu})\Phi(z)\,, \tag{3.81}$$

from where we read out the generators that act on the Hilbert space

$$\begin{cases} P_\mu = i\partial_\mu\,, \\ \mathcal{D} = i(\hat{\boldsymbol{\ell}} + z\partial)\,, \\ L_{\mu\nu} = i(z_\mu\partial_\nu - z_\nu\partial_\mu - S_{\mu\nu})\,, \\ K_\mu = -i(2z_\mu(z\partial) - z^2\partial_\mu + 2\hat{\boldsymbol{\ell}}z_\mu - 2z^\nu S_{\mu\nu})\,, \end{cases} \tag{3.82}$$

as well as the subsequent infinitesimal conformal transformations

$$\begin{cases} \delta_P\Phi(z) = -a^\mu\partial_\mu\Phi(z)\,, \\ \delta_{\mathcal{D}}\Phi(z) = -\alpha(\hat{\boldsymbol{\ell}} + z\partial)\Phi(z)\,, \\ \delta_L\Phi(z) = -\widetilde{b}^{\nu\mu}(z_\mu\partial_\nu - z_\nu\partial_\mu - S_{\mu\nu})\Phi(z)\,, \\ \delta_K\Phi(z) = c^\mu(2z_\mu(z\partial) - z^2\partial_\mu + 2\hat{\boldsymbol{\ell}}z_\mu - 2z^\nu S_{\mu\nu})\Phi(z)\,. \end{cases} \tag{3.83}$$

[3]Representations of generators depend on the space on which these operators are acting on. Previously we were studying the conformal group action on coordinates, this is the space of interest was $\mathbb{R}^D$ for a D-dimensional spacetime. Now we are studying actions on QFT operators, this is on the Hilbert space and, as a result, some (or all) generators may have a different form, a different representation. Nevertheless, the procedure that we will follow to find the representation for the Hilbert space will respect the algebra (3.56).

[4]Hausdorff formula for two operators A and B: $e^A B e^{-A} = e^{[A,\cdot]}B = B + [A, B] + \frac{1}{2}[A, [A, B]] + \cdots$.

We can make use of the above infinitesimal transformations to induce their finite structure by exponential mapping and the relation (3.65):

$$\text{Translations: } \Phi'(z') = \Phi(z)\,, \tag{3.84}$$

$$\text{Scale transformation: } \Phi'(\lambda z) = \lambda^{-\ell}\Phi(z)\,, \tag{3.85}$$

$$\text{Lorentz transf. for scalar: } \phi'(z') = \phi(z)\,, \tag{3.86}$$

$$\text{Lorentz transf. for vector: } A'^{\rho}(z') = R^{\rho}{}_{\alpha}A^{\alpha}(z)\,, \tag{3.87}$$

$$\text{Lorentz transf. for fermion: } \psi'(z') = \exp\left\{\frac{i}{2}\widetilde{b}^{\nu\mu}\sigma_{\mu\nu}\right\}\psi(z)\,, \tag{3.88}$$

$$\text{Inversion for scalar: } (\mathcal{I}\phi)(1/z) = (z^2)^{\ell}\phi(z)\,, \tag{3.89}$$

$$\text{Inversion for vector: } (\mathcal{I}A)^{\rho}(1/z) = (z^2)^{\ell}\mathcal{I}^{\rho}_{\mu}(z)A^{\mu}(z)\,, \tag{3.90}$$

$$\text{Inversion for fermion: } (\mathcal{I}\psi)(1/z) = (z^2)^{\ell}(\gamma\hat{z})\psi(z)\,, \tag{3.91}$$

where $\hat{z}^{\mu} = z^{\mu}/\sqrt{z^2}$, λ is a constant (parameter for dilations), ℓ is the scaling dimension of the corresponding field and z' is the corresponding transformed position according to Sect. 3.3. For more details on the inversion transformation vid. App. F as well as Refs. [105, 106].

Finally, from the inversion of vector operators, one can induce that of $n-$ranked tensors:

$$(\mathcal{I}\mathcal{O}_{\mu_1\mu_2\cdots\mu_n})(1/z) = (z^2)^{\ell}\mathcal{I}^{\nu_1}_{\mu_1}(z)\mathcal{I}^{\nu_2}_{\mu_2}(z)\cdots\mathcal{I}^{\nu_n}_{\mu_n}(z)\mathcal{O}_{\nu_1\nu_2\cdots\nu_n}(z)\,. \tag{3.92}$$

3.3.4 Conformal vector and scalar

For three points z_1, z_2, z_3 we can define the vector[5]

$$\Lambda^{\mu}(z_1,z_2,z_3) = \frac{1}{2}\partial_3^{\mu}\ln\frac{(z_2-z_3)^2}{(z_1-z_3)^2} = \frac{(z_1-z_3)^{\mu}}{(z_1-z_3)^2} - \frac{(z_2-z_3)^{\mu}}{(z_2-z_3)^2}\,, \qquad \partial_3^{\mu} = \frac{\partial}{\partial z_{3,\mu}}\,. \tag{3.93}$$

This object is a *conformal vector* with scaling dimension one for z_3 and zero for z_1, z_2 in the sense that it transforms as a vector at z_3 under inversions and conformal transformations, this is

$$\Lambda^{\mu}(\mathcal{I}z_1,\mathcal{I}z_2,\mathcal{I}z_3) = z_3^2\mathcal{I}^{\mu}_{\rho}(z_3)\Lambda^{\rho}(z_1,z_2,z_3)\,, \tag{3.94}$$

$$\Lambda^{\mu}(z_1',z_2',z_3') = \Omega(z_3)R^{\mu}{}_{\rho}\Lambda^{\rho}(z_1,z_2,z_3)\,. \tag{3.95}$$

One can also define a scalar function $Z(z_1,z_2,z_3)$ as

$$Z(z_1,z_2,z_3) = \left(\frac{(z_1-z_2)^2}{(z_1-z_3)^2(z_2-z_3)^2}\right)^{T/2}\,, \quad T\in\mathbb{R}\,, \tag{3.96}$$

[5]The order in which points z_1, z_2, z_3 appear in the argument of Λ^{μ} matters.

which under inversions and conformal transformations behaves as a conformal scalar at z_3 with scaling dimension T:

$$Z(\mathcal{I}z_1, \mathcal{I}z_2, \mathcal{I}z_3) = (z_3^2)^T Z(z_1, z_2, z_3)\,, \tag{3.97}$$

$$Z(z_1', z_2', z_3') = \Omega(z_3)^T Z(z_1, z_2, z_3)\,, \tag{3.98}$$

where it was used

$$(z_1' - z_2')^2 = \Omega(z_1)^{-1}\Omega(z_2)^{-1}(z_1 - z_2)^2\,. \tag{3.99}$$

These conformal vectors and scalars will play a key role for the computation of correlators in section 3.3.5 and for the conformal operator-product expansion that we will address in section 3.4.

3.3.5 Two- and three-point correlators

Classically, the invariance of the action $S[\Phi]$ under a continuous transformation implies the existence of a preserved charge or, in other words, of a conserved current (Noether theorem). However, in QFTs the object that determines the dynamics is not only the action itself but also the correlators between fields. For these ones, transformations (that could be symmetries of the action from a classical point of view) establish relations among different correlators. The $n-$point correlator or Green function is defined as the vacuum expectation value of the product of n fields

$$\langle 0|\Phi(z_1)\cdots\Phi(z_n)|0\rangle \equiv \langle\Phi(z_1)\cdots\Phi(z_n)\rangle = \frac{1}{\mathcal{N}_{\text{vac}}}\int D\Phi\,\Phi(z_1)\cdots\Phi(z_n)e^{iS[\Phi]}\,, \tag{3.100}$$

with $\mathcal{N}_{\text{vac}}$ the vacuum functional. From this expression and a transformation $z \to z'$ such that $\Phi(z) \to \Phi'(z') = \mathcal{F}(\Phi(z))$ [100],

$$\begin{aligned}\langle\Phi(z_1')\cdots\Phi(z_n')\rangle &= \frac{1}{\mathcal{N}_{\text{vac}}}\int D\Phi\,\Phi(z_1')\cdots\Phi(z_n')e^{iS[\Phi]}\\ &= \frac{1}{\mathcal{N}_{\text{vac}}}\int D\Phi'\,\Phi'(z_1')\cdots\Phi'(z_n')e^{iS[\Phi']}\\ &= \langle\Phi'(z_1')\cdots\Phi'(z_n')\rangle\,,\end{aligned} \tag{3.101}$$

where we just introduced a name change to the integral variable, i.e. $\Phi \to \Phi'$. Now, if we consider the functional change $\Phi'(z') = \mathcal{F}(\Phi(z))$ and hypothesize that the Jacobian of the integral measure cancels the variation of the action, this is

$$\left|\det\left(\frac{D\Phi'}{D\Phi}\right)\right| = e^{-i(S[\mathcal{F}(\Phi)]-S[\Phi])}\,, \tag{3.102}$$

or that it is a constant and therefore cancels with the same factor in $\mathcal{N}_{\text{vac}}$, then it holds:

$$\begin{aligned}\langle\Phi'(z_1')\cdots\Phi'(z_n')\rangle &= \frac{1}{Z}\int D\Phi\,\mathcal{F}(\Phi(z_1))\cdots\mathcal{F}(\Phi(z_n))e^{iS[\Phi]}\\ &= \langle\mathcal{F}(\Phi(z_1))\cdots\mathcal{F}(\Phi(z_n))\rangle\,.\end{aligned} \tag{3.103}$$

In such a case, it is true that

$$\langle\Phi(z_1')\cdots\Phi(z_n')\rangle = \langle\mathcal{F}(\Phi(z_1))\cdots\mathcal{F}(\Phi(z_n))\rangle\,. \tag{3.104}$$

Henceforth we will assume the validity of this last equation when computing any correlator. A detailed calculation of the two-point Green function for scalar fields can be found in App. G. The final result is that for a non-zero separation ($z_{12} = z_1 - z_2 \neq 0$):

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \begin{cases} \dfrac{c_{12}}{(z_{12}^2)^{\ell_1}}\,, & \text{if } \ell_1 = \ell_2\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.105}$$

Note that we can always choose an orthogonal field basis such that $c_{ij} = \delta_{ij}$.

For vectors and spinors, following the same procedure as for scalar fields to make correlators transform accordingly to the conformal group, we have the expressions ($z_{12} \neq 0$):

$$\langle A_1^\mu(z_1)A_2^\nu(z_2)\rangle = \begin{cases} c_{12}\dfrac{\mathcal{I}^{\mu\nu}(z_1 - z_2)}{(z_{12}^2)^{\ell_1}}\,, & \text{if } \ell_1 = \ell_2\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{3.106}$$

and

$$\langle \psi_1(z_1)\psi_2(z_2)\rangle = \begin{cases} c_{12}\dfrac{\gamma^\rho(\hat{z}_1 - \hat{z}_2)_\rho}{(z_{12}^2)^{\ell_1}}\,, & \text{if } \ell_1 = \ell_2\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.107}$$

Clearly, the constant c_{12} in scalar, vector and spinor cases is different in general.

The formulas exhibited thus far hold as long as $z_{12} \neq 0$. For the null case, $z_{12} = 0$, with two operators whose scaling dimensions satisfy to be in the relation $\ell_2 = D - \ell_1$ with D the spacetime dimension, the vacuum expectation value is given by

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = c_{12}\delta(z_{12})\,, \quad \text{if } \ell_2 = D - \ell_1\,, \tag{3.108}$$

$$\langle A_1^\mu(z_1)A_2^\nu(z_2)\rangle = c_{12}\delta(z_{12})g^{\mu\nu}\,, \quad \text{if } \ell_2 = D - \ell_1\,, \tag{3.109}$$

and for higher-ranked tensors in irreps of the Lorentz group (traceless operators),

$$\langle \mathcal{O}_1^{\mu_1\cdots\mu_n}(z_1)\mathcal{O}_2^{\nu_1\cdots\nu_n}(z_2)\rangle = \begin{cases} c_{12}\dfrac{1}{(z_{12}^2)^{\ell_1}}\mathcal{I}^{\mu_1\nu_1}(z_{12})\cdots\mathcal{I}^{\mu_n\nu_n}(z_{12}) - \text{traces}\,, & \text{if } \ell_1 = \ell_2\,, \\ c_{12}\delta(z_{12})\mathscr{C}^{\mu_1\cdots\mu_n;\nu_1\cdots\nu_n}\,, \quad \text{if } \ell_2 = D - \ell_1\,, \\ 0\,, \quad \text{otherwise}\,. \end{cases} \tag{3.110}$$

Indices μ_i and ν_j in both sides of Eq. (3.110) must have the same symmetry, so the projector on the appropriate symmetry class is considered implicit hereafter unless deemed necessary. Note that $\mathscr{C}$ is a traceless tensor built out of Minkowsky metrics with the symmetry inherited from the tensor operators $\mathcal{O}_1$ and $\mathcal{O}_2$. $\mathscr{C}$ can only be constructed with metrics because it should not modify the conformal covariance of the Dirac delta.

Nevertheless, if apart from being traceless, we also consider the operators in the correlator as symmetric (highest-spin Lorentz irrep), then $\mathscr{C}$ will not only be traceless but also symmetric under the exchange of μ_i and ν_j indices separately, i.e.

$$\mathscr{C}^{\mu_1\cdots\mu_n;\nu_1\cdots\nu_n} = \mathscr{C}^{(\mu_1\cdots\mu_n);(\nu_1\cdots\nu_n)}\,. \tag{3.111}$$

As a rule, $(\mu_1\cdots\mu_n)$ stands for symmetrization and normalization by $n!$. Likewise, $[\mu_1\cdots\mu_n]$ represents antisymmetrization and division by $n!$. For instance, for the case $n = 2$ we have

$$\mathscr{C}^{\mu_1\mu_2;\nu_1\nu_2} = \frac{1}{2}(g^{\mu_1\nu_1}g^{\mu_2\nu_2} + g^{\mu_1\nu_2}g^{\mu_2\nu_1}) - \frac{1}{D}g^{\mu_1\mu_2}g^{\nu_1\nu_2}\,, \tag{3.112}$$

which is the projector onto the space of symmetric and traceless tensors: $\mathscr{C}^{\nu_1\nu_2}_{\mu_1\mu_2}\mathcal{O}_{\nu_1\nu_2} = \mathcal{O}_{(\mu_1\mu_2)} - (1/D)g_{\mu_1\mu_2}\mathcal{O}^{\lambda}_{\ \lambda}\,.$

It is also important to notice that for two tensors of different rank, their correlator vanishes:

$$\langle \mathcal{O}_1^{\mu_1\cdots\mu_n}(z_1)\mathcal{O}_2^{\nu_1\cdots\nu_m}(z_2)\rangle = 0\,, \quad \text{if } n \neq m \text{ and/or } \ell_1 \neq \ell_2\,. \tag{3.113}$$

For the proof cf. App. H and Refs. [107, 108].

To conclude this section, we can examine the three-point correlation function involving one tensor of rank and spin n in a Lorentz irrep (symmetric and traceless) and two scalars that turns out to be,

$$\begin{aligned}\langle A(z_1)B(z_2)\mathcal{O}_{\alpha_1\cdots\alpha_n}(z_3)\rangle = &\frac{c_{ABn}}{(z_{12}^2)^{(-t_n+\ell_A+\ell_B)/2}(z_{23}^2)^{(t_n+\ell_B-\ell_A)/2}(z_{13}^2)^{(t_n+\ell_A-\ell_B)/2}} \\ &\times \Lambda_{\alpha_1}(z_1,z_2,z_3)\cdots\Lambda_{\alpha_n}(z_1,z_2,z_3) - \text{traces}\,,\end{aligned} \tag{3.114}$$

where $c_{ABn} \in \mathbb{C}$. The *conformal twist* $t_n = \ell_n - n$ has been introduced[6] by means of the tensor scaling dimension ℓ_n. The tensor structure of the correlator has been reduced to a (symmetric) product of conformal vectors Λ_α, vid. Eq. (3.93), that behave appropriately under conformal transformations and inversions allowing for a conformally covariant vacuum expectation value.

Shadow-operator formalism

The shadow-operator formalism is a methodology that allows you to isolate the contribution of a particular operator to the OPE. To understand it, we first notice the existence of the following operator that collapse the OPE onto the contribution of an operator $\mathcal{O}$ with scaling dimension ℓ_n and rank n [105]:

$$\mathscr{P}_{\ell_n,n} = \int d^Dz\, \widetilde{\mathcal{O}}_{\alpha_1\cdots\alpha_n}(z)|0\rangle\langle 0|\mathcal{O}^{\alpha_1\cdots\alpha_n}(z)\,. \tag{3.115}$$

Here, $\widetilde{\mathcal{O}}$ is the *shadow operator* of $\mathcal{O}$ with scaling dimension $\widetilde{\ell}_n = D - \ell_n$ for a D-dimensional spacetime. Operators $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are normalized in such a way that $c_{12} = 1$ in the second equation of (3.110). Indeed, the operator $\mathscr{P}_{\ell_n,n}$ satisfies the conditions of a projector:

$$\mathscr{P}_{\ell_n,n}\mathscr{P}_{\ell'_m,m} = \delta_{\ell_n\ell'_m}\delta_{nm}\mathscr{P}_{\ell_n,n}\,, \tag{3.116}$$

$$\sum_{\ell_n,n}\mathscr{P}_{\ell_n,n} = 1\,. \tag{3.117}$$

The first property, with δ_{ij} as the Kronecker delta, comes from Eqs. (3.110) and (3.113). The last one is a consequence of the conformal Casimir operators depending

[6]This conformal twist reduces to the geometric twist seen in Sect. 3.1 when the operator is symmetric, so its spin representation coincides with its rank, and scaling dimension reduces to the classical dimension in energy units of the operator. By "classical" we mean without anomalous contributions coming from the renormalization procedure in QFTs.

on the tensor rank n and the scaling dimension ℓ_n [107]:

$$C_I = -2\ell_n(D-\ell_n) + 2n(n+2)\,, \tag{3.118}$$
$$C_{II} = n(n+2)[3-\ell_n(D-\ell_n)]\,. \tag{3.119}$$

Since the conformal algebra is of rank 3, you can build a third Casimir operator but this one turns out to be zero. Note that under the exchange $\ell_n \leftrightarrow D-\ell_n$ the Casimir operators do not get modified, implying that the representation based on operators is equivalent to the one based on shadow operators. As a result all possible operators can be labelled with only two numbers: rank and scaling dimension. If we sum over all projectors built with operators of any rank and scaling dimension, then we map the space of operators completely and (3.117) must hold.

Omitting possible Lorentz indices, for any set of operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ and the projector (3.115) we can write

$$\begin{aligned}\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\rangle &= \sum_{\ell_n,n} \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathscr{P}_{\ell_n,n}\mathcal{O}_3(z_3)\rangle \\ &= \sum_{\ell_n,n}\int d^Dz\,\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\widetilde{\mathcal{O}}_{\beta_1\cdots\beta_n}(z)\rangle\langle \mathcal{O}^{\beta_1\cdots\beta_n}(z)\mathcal{O}_3(z_3)\rangle \\ &= \langle 0|\sum_{\ell_n,n}\int d^Dz\,\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\widetilde{\mathcal{O}}_{\beta_1\cdots\beta_n}(z)\rangle \mathcal{O}^{\beta_1\cdots\beta_n}(z)\mathcal{O}_3(z_3)|0\rangle\,,\end{aligned} \tag{3.120}$$

which may be rearranged as

$$\langle 0|\left[\mathcal{O}_1(z_1)\mathcal{O}_2(z_2) - \sum_{\ell_n,n}\int d^Dz\,\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\widetilde{\mathcal{O}}_{\beta_1\cdots\beta_n}(z)\rangle \mathcal{O}^{\beta_1\cdots\beta_n}(z)\right]\mathcal{O}_3(z_3)|0\rangle = 0\,. \tag{3.121}$$

Since this last expression must hold for any conformal operator $\mathcal{O}_3$ we may conclude that the term in square brackets is zero and provides the *conformal operator-product expansion* (COPE):

$$\mathcal{O}_1(z_1)\mathcal{O}_2(z_2) = \sum_{\ell_n,n}\int d^Dz\,\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\widetilde{\mathcal{O}}_{\beta_1\cdots\beta_n}(z)\rangle \mathcal{O}^{\beta_1\cdots\beta_n}(z)\,. \tag{3.122}$$

The vacuum expectation value in the integrand is usually called the *shadow triangle* in accordance with the pictorial representation proposed by S. Ferrara and G. Parisi [109]. This expectation value can be related to Wilson coefficients.

By means of Eqs. (3.72) and (E.2),

$$\mathcal{O}^{\beta_1\cdots\beta_n}(z) = e^{z\partial_y}\mathcal{O}^{\beta_1\cdots\beta_n}(y)\Big|_{y=0}\,, \quad \partial^\mu_y = \frac{\partial}{\partial y_\mu}\,, \tag{3.123}$$

the formula for the COPE becomes

$$\begin{aligned}\mathcal{O}_1(z_1)\mathcal{O}_2(z_2) &= \sum_{\ell_n,n}\int d^Dz\,\left[\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\widetilde{\mathcal{O}}_{\beta_1\cdots\beta_n}(z)\rangle e^{z\partial_y}\right]\mathcal{O}^{\beta_1\cdots\beta_n}(y)\Big|_{y=0} \\ &= \sum_{\ell_n,n}\int d^Dz\,\left[\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\widetilde{\mathcal{O}}_{\beta_1\cdots\beta_n}(z)\rangle e^{izr}\right]\mathcal{O}^{\beta_1\cdots\beta_n}(y)\Big|_{y=0}\,,\end{aligned} \tag{3.124}$$

where the change of notation $-i\partial_y^\mu \to r^\mu$ has been used. This way, the problem of computing the COPE has been simplified to the Fourier transform of the shadow triangle, whose form can be fully predicted up to a constant thanks to the constraints imposed by the requirement of conformal covariance/invariance as seen before. This last formula constitute the grounds for the next section where the calculation of correlators and the OPE employing the techniques of CFT will be applied to the product of two quark currents in QCD.

3.4 Conformal operator-product expansion for two currents

As already explained in Chs. 1 and 2, exclusive processes such as DVCS, TCS and DDVCS are different types of Compton scattering. The photon-hadron interaction can be described in general by the reaction:

$$\gamma^{(*)}(q) + N(p) \to \gamma^{(*)}(q') + N(p')\,, \tag{3.125}$$

where the hadron N absorbs and emits (virtual) photons with momenta q and q', respectively. Of course, the momentum transfer to the hadron is non-zero: $p \neq p'$.

In QCD, the reaction (3.125) is described by the Compton tensor, previously introduced in Eq. (1.60). In Ch. 1 this tensor was proven to be mostly dominated by the light-cone region of the spacetime, $z^2 = 0$, a feature closely related to the limit of infinite photon virtualities. However, this tensor receives subleading contributions from spacetime positions outside the light-cone. To address these contributions one needs to relax the conditions on the photon virtualities, i.e. to consider them finite yet large enough to allow for a perturbative expansion. Such an expansion around the light-cone should be reproducible as an OPE of the form

$$j^\mu(z_1)j^\nu(z_2) \sim \sum_j C_j(z_{12},\partial)\mathcal{O}_j\left(\frac{z_1+z_2}{2}\right)\,, \tag{3.126}$$

where $z_{12} = z_1 - z_2$ and $\{\mathcal{O}_j\}$ is some basis of operators with the same quantum numbers as the LHS. A formal way to address this expansion is to apply the COPE introduced in Eq. (3.124) to the spin-1 electromagnetic quark currents of Eq. (1.18), that is:

$$j^\mu(z_1)j^\nu(z_2) = \sum_{\widetilde{\ell}_N,N} \int d^Dz_3 \left[\langle j^\mu(z_1)j^\nu(z_2)\widetilde{\mathcal{O}}^{\alpha_1\cdots\alpha_N}(z_3)\rangle e^{iz_3r}\right] \mathcal{O}_{\alpha_1\cdots\alpha_N}(y)\Big|_{y=0}\,. \tag{3.127}$$

Therefore, we need to solve the Fourier transform of the shadow triangle:

$$T^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,r) = \int d^Dz\, \langle j^\mu(z_1)j^\nu(z_2)\widetilde{\mathcal{O}}^{\alpha_1\cdots\alpha_N}(z_3)\rangle e^{iz_3r}\,. \tag{3.128}$$

Remember that $r^\mu \to -i\partial_y^\mu$, hence the above tensor provides the coefficients C_j of Eq. (3.126).

We want to introduce corrections to the kinematic LT described by the usual GPDs and CFFs. GPDs are matrix elements of geometric twist-2 operators, as proven in App. D, hence symmetric and traceless. Therefore, the subset of symmetric and traceless operators with the same quantum numbers as the currents is, in momentum space, related to GPDs and their derivatives. Therefore, the COPE restricted to this kind of operators will correspond, in momentum space, to an expansion in kinematic power corrections $\sim |t|/Q^2$ and $\sim M^2/Q^2$ with the hadron structure still represented by the GPDs that appear at LT. These corrections are the effects we are interested in. Other components of the COPE are related to higher-twist GPDs, this is new non-perturbative distributions that are expected to enter the amplitude by powers of $\sim \Lambda^2_{\rm QCD}/Q^2$ where $\Lambda_{\rm QCD}$ is the QCD Landau-pole. The dumping factor $\Lambda^2_{\rm QCD}/Q^2$ is of order 0.04, whereas the finite-t and target-mass corrections in facilities such as JLab are typically of order 10% to 30%. For this reason, the kinematic corrections are the main focus of this chapter.

Since the conformal operators we are interested in will be connected to GPDs, they will basically consist of derivatives (N for an $N-$ranked tensor operator) acting on a bilinear $\bar{q}\Gamma q$, where Γ is made out of Dirac-gamma matrices. Consequently, the scaling dimension of the LT conformal operator $\mathcal{O}^{\alpha_1\cdots\alpha_N}$ is

$$\ell_N = D - 2 + N + \gamma_N\,, \tag{3.129}$$

where γ_N is the anomalous dimension which must be taken at the Fisher-Wilson fixed point discussed in Sect. 3.2. At LO, $\gamma_N = 0$. Likewise, the scaling dimension of the corresponding shadow operator $\widetilde{\mathcal{O}}^{\alpha_1\cdots\alpha_N}$ is

$$\widetilde{\ell}_N = 2 - N - \gamma_N\,. \tag{3.130}$$

Because $\widetilde{\ell}_N$ is uniquely determined by the spin N of the LT conformal operator, the sum on Eq. (3.127) can be simplified to a sum over N only.

Following the procedure of Sect. 3.3.5, the shadow triangle in Eq. (3.128) can be written by means of the inversion tensor $\mathcal{I}_{\mu\nu}$ (3.64), the conformal vectors Λ^μ (3.93) and the conformal scalar Z (3.96). For vector operators $\widetilde{\mathcal{O}}$, a total of five structures compatible with conformal covariance are found [46]. Taking into account that the currents are bosonic operators, they commute and the triangle is symmetric under the exchange $(z_1,\mu) \leftrightarrow (z_2,\nu)$. Then, it is possible to reduce the number of structures to four for even N and just one for odd N:

$$\langle j^\mu(z_1) j^\nu(z_2) \widetilde{\mathcal{O}}^{\alpha_1\cdots\alpha_N}(z_3)\rangle = \sum_{i=0}^{3} \mathbb{C}_i \mathbb{T}_i^{\mu\nu\alpha_1\cdots\alpha_N}\,, \tag{3.131}$$

where

$$\begin{aligned}
\mathbb{T}_0^{\mu\nu\alpha_1\cdots\alpha_N} &= \frac{1}{(z_{12}^2)^{D-1}} \mathcal{I}^{\mu\nu}(z_{12})\, Z(z_1,z_2,z_3)\, \Lambda^{\alpha_1\cdots\alpha_N}(z_1,z_2,z_3)\,,\\
\mathbb{T}_1^{\mu\nu\alpha_1\cdots\alpha_N} &= \frac{1}{(z_{12}^2)^{D-2}} \partial_1^\mu \partial_2^\nu \left(Z(z_1,z_2,z_3)\, \Lambda^{\alpha_1\cdots\alpha_N}(z_1,z_2,z_3)\right),\\
\mathbb{T}_2^{\mu\nu\alpha_1\cdots\alpha_N} &= \frac{1}{(z_{12}^2)^{D-2}} \mathcal{I}^\mu_\rho(z_{12}) \mathcal{I}^\nu_\sigma(z_{12}) \partial_1^\sigma \partial_2^\rho \left(Z(z_1,z_2,z_3)\, \Lambda^{\alpha_1\cdots\alpha_N}(z_1,z_2,z_3)\right),\\
\mathbb{T}_3^{\mu\nu\alpha_1\cdots\alpha_N} &= \frac{1}{(z_{12}^2)^{D-2}} \Big[\partial_1^\mu \left(\Lambda^{\alpha_1\cdots\alpha_N}(z_1,z_2,z_3)\, \partial_2^\nu Z(z_1,z_2,z_3)\right)\\
&\qquad + \partial_2^\nu \left(\Lambda^{\alpha_1\cdots\alpha_N}(z_1,z_2,z_3)\, \partial_1^\mu Z(z_1,z_2,z_3)\right)\Big]\,,
\end{aligned} \tag{3.132}$$

where $\partial_i^\mu = \partial/\partial z_{i,\mu}$ with $i \in \{1,2,3\}$ and for simplicity

$$\Lambda^{\alpha_1\cdots\alpha_N}(z_1,z_2,z_3) = \Lambda^{\alpha_1}(z_1,z_2,z_3) \times \cdots \times \Lambda^{\alpha_N}(z_1,z_2,z_3)\,. \tag{3.133}$$

Symmetrization and tracelessness over $\alpha_1, \cdots, \alpha_N$ indices is supposed implicit and we used the conformal vector and scalar introduced earlier in Eqs. (3.93) and (3.96), respectively. These four structures contribute for even N, while for odd N we are left with only one, as stated before.

Including current conservation, $\partial_\mu j^\mu = 0$ and so $U(1)$-electromagnetic gauge invariance, reduces the number of structures even further: two for even N and zero for odd N. Hence, for even N,

$$\langle j^\mu(z_1) j^\nu(z_2) \widetilde{\mathcal{O}}_N^{\alpha_1\cdots\alpha_N}(z_3)\rangle = c_1^N \mathcal{A}_1^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3) + c_2^N \mathcal{A}_2^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3)\,, \tag{3.134}$$

where c_1^N and c_2^N are combinations of $\mathbb{C}_0$, ..., $\mathbb{C}_3$ and

$$\begin{aligned}
\mathcal{A}_1^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3) &= a_{10}^N\, \mathbb{T}_0^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3) - \mathbb{T}_2^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3)\,,\\
\mathcal{A}_2^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3) &= \mathbb{T}_3^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3) + a_{21}^N\, \mathbb{T}_1^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3)\\
&\quad + a_{20}^N\, \mathbb{T}_0^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3)\,.
\end{aligned} \tag{3.135}$$

The coefficients a_{10}^N, a_{21}^N and a_{20}^N are given in terms of the spacetime dimension D, the scaling dimension $\widetilde{\ell}_N$ of the shadow operator $\widetilde{\mathcal{O}}$ and its spin N. Explicit expressions are collected in Ref. [47].

The form of the above tensors (3.132) reflects their connection to the shadow triangle for scalar currents, vid. Eq. (3.114). Hence, this triangle takes the role as the generator of the Fourier transforms of the structures in (3.132).

Taking into account that we are interested in geometric twist-2 operators, which are symmetric and traceless, we can consider the contraction of said triangle with the tensor $n_{\alpha_1}\cdots n_{\alpha_N}$. The n vectors correspond to the the usual lightlike vector which projects the positive-longitudinal light-cone coordinate, in the same way as it was introduced in Chs. 1 and 2. The advantage of this contraction is clear: because $n^2 = 0$, the traces (proportional to the metric) of the triangle vanish identically simplifying the calculation.

Because $n_{\alpha_1}\cdots n_{\alpha_N}$ has the same features as the basis of operators used in the COPE (3.127), then after the calculation, we can put the triangle back in the COPE by simply removing all the auxiliary n vectors. Thereby, we require:

$$\begin{aligned}
S(z_1,z_2,r|\ell_A,\ell_B,\widetilde{\ell}_N) &= \int d^D z_3\, n^{\alpha_1}\cdots n^{\alpha_N}\langle A(z_1)B(z_2)\widetilde{\mathcal{O}}_{\alpha_1\cdots\alpha_N}(z_3)\rangle e^{iz_3 r}\\
&= \int d^D z_3\, \frac{\widetilde{c}_{ABN}\,(n\Lambda(z_1,z_2,z_3))^N\, e^{iz_3 r}}{(z_{12}^2)^{(-t_N+\ell_A+\ell_B)/2}(z_{23}^2)^{(t_N+\ell_B-\ell_A)/2}(z_{13}^2)^{(t_N+\ell_A-\ell_B)/2}}\,,
\end{aligned} \tag{3.136}$$

where $\widetilde{c}_{ABN} \in \mathbb{C}$ is a constant, while ℓ_A, ℓ_B and $\widetilde{\ell}_N = t_N + N$ are the scaling dimensions of the operators involved: two scalar fields A and B, and the set of geometric twist-2 shadow operators $\widetilde{\mathcal{O}}$ that serves as basis for the COPE, respectively. The Fourier transform of the scalar shadow triangle (3.136) was calculated, in Euclidean spacetime, in Refs. [46, 110]:

$$
\begin{aligned}
&S(z_1, z_2, r|\ell_A, \ell_B, \widetilde{\ell}_N) = \\
&= \frac{\mathcal{N}}{|z_{12}|^{\ell_A+\ell_B-t_N}} \sum_{k=0}^{N} \frac{\Gamma(2\varkappa_N - k)}{(N-k)!} (z_{12}\cdot n)^{N-k} [-i(r\cdot n) z_{12}^2]^k \\
&\times \int_0^1 du\, (u\bar{u})^{N-1+\frac{1}{2}t_N} \left(\frac{u}{\bar{u}}\right)^{\ell_{AB}} P_k^{(\varkappa_N-\ell_{AB}-k-\frac{1}{2},\, \varkappa_N+\ell_{AB}-k-\frac{1}{2})}(2u-1) \\
&\times \mathbf{I}_{\ell_N+k-D/2}\left(\sqrt{u\bar{u}z_{12}^2 r^2}\right) e^{iz_{21}^u r}\,,
\end{aligned}
\tag{3.137}
$$

where

$$
\bar{u} = 1-u\,, \quad z_{21}^u = \bar{u} z_2 + u z_1\,, \tag{3.138}
$$

$$
\ell_{AB} = \frac{1}{2}(\ell_A - \ell_B)\,, \quad t_N = \ell_N - N\,, \quad \varkappa_N = \frac{1}{2}(D - t_N - 1)\,, \tag{3.139}
$$

and

$$
\mathcal{N} = \widetilde{c}_{ABN} \frac{(2\pi)^{D/2} N!}{2^{\widetilde{\ell}_N-1}\Gamma(\widetilde{\ell}_N - 1)\Gamma(\varkappa_N - \ell_{AB} + \frac{1}{2})\Gamma(\varkappa_N + \ell_{AB} + \frac{1}{2})}\,. \tag{3.140}
$$

Also, $P_N^{(a,b)}(z)$ are Jacobi's polynomials and

$$
\mathbf{I}_\nu(z) = 2^{-\nu} \sum_{m=0}^{\infty} \frac{(z^2)^m}{2^{2m} m! \Gamma(\nu+m+1)}\,. \tag{3.141}
$$

Making use of this formula, one can express the Fourier transforms of the structures (3.132, 3.135) as (N is even):

$$
\begin{aligned}
\mathbb{T}_0^{\mu\nu}(z_1,z_2,r) &= \frac{1}{(z_{12}^2)^{D-1}} \mathcal{I}^{\mu\nu}(z_{12})\, S(z_1,z_2,r|0,0,\widetilde{\ell}_N)\,, \\
\mathbb{T}_1^{\mu\nu}(z_1,z_2,r) &= \frac{1}{(z_{12}^2)^{D-2}} \partial_1^\mu \partial_2^\nu\, S(z_1,z_2,r|0,0,\widetilde{\ell}_N)\,, \\
\mathbb{T}_2^{\mu\nu}(z_1,z_2,r) &= \frac{1}{(z_{12}^2)^{D-2}} \mathcal{I}_\rho^\mu(z_{12}) \mathcal{I}_\sigma^\nu(z_{12}) \partial_1^\sigma \partial_2^\rho\, S(z_1,z_2,r|0,0,\widetilde{\ell}_N)\,, \\
\mathbb{T}_3^{\mu\nu}(z_1,z_2,r) &= \frac{\widetilde{\ell}_N - N}{(z_{12}^2)^{D-2}} \Big\{ \left(\partial_1^\mu z_{21}^\nu + \partial_2^\nu z_{12}^\mu\right) \frac{1}{z_{12}^2} S(z_1,z_2,r|0,0,\widetilde{\ell}_N) \\
&\quad - \partial_2^\nu (z_1^\mu + i\partial_r^\mu) S(z_1,z_2,r|0,0,\widetilde{\ell}_N+1) - \partial_1^\mu (z_2^\nu + i\partial_r^\nu) S(z_2,z_1,r|0,0,\widetilde{\ell}_N+1) \Big\}\,,
\end{aligned}
\tag{3.142}
$$

where $\partial_r^\mu = \partial/\partial r_\mu$ and

$$
\mathbb{T}_i^{\mu\nu}(z_1,z_2,r) = \int d^D z_3\, n_{\alpha_1} \cdots n_{\alpha_N} \mathbb{T}_i^{\mu\nu\alpha_1\cdots\alpha_N}(z_1,z_2,z_3) e^{iz_3 r}\,. \tag{3.143}
$$

Introducing these expressions in Eqs. (3.127), removing the n vectors and redefining $\mathbb{C}_i \times \widetilde{c}_{ABN} \to \mathbb{C}_i$, the COPE for two conserved currents of spin 1 in the basis of symmetric and traceless conformal vector operators takes the form:

$$
\begin{aligned}
j^\mu(z_1)j^\nu(z_2) = & \frac{1}{(z_{12}^2)^{D-2}} \sum_{N,\,\mathrm{even}} 2^{\lambda_N}\Gamma(\lambda_N+1)\int_0^1 du\,(u\bar{u})^{j_N-1}\sum_{k=0}^{N}\frac{N!\,\Gamma(\varkappa_N-k)}{(N-k)}\Gamma(\varkappa_N) \\
& \times \Bigg\{ \mathbf{D}_N^{\mu\nu}(z_{12}^2)^{\tau_N}\left(\frac{z_{12}^2}{4}\right)^k C_k^{\varkappa_N-k}(u-\bar{u})\,\mathbf{I}_{\lambda_N+k}\Big(\sqrt{-u\bar{u}z_{12}^2\partial^2}\Big)\mathcal{O}_N^{(k)}(z_{21}^u) \\
& - \widetilde{\mathbb{C}}_3^N\frac{(\widetilde{\ell}_N-N)(\varkappa_N-k)}{(N-2\varkappa_N)(\varkappa_N+\frac{1}{2})}\frac{1}{\bar{u}}\Big[C_k^{(1+\varkappa_N-k)}(u-\bar{u})-C_{k-1}^{(1+\varkappa_N-k)}(u-\bar{u})\Big] \\
& \times \Bigg[\partial_2^\nu(z_{12}^2)^{\tau_N-1}\left(\frac{z_{12}^2}{4}\right)^k\Big(z_1^\mu\,\mathbf{I}_{\lambda_N+k-1}\Big(\sqrt{-u\bar{u}z_{12}^2\partial^2}\Big)\mathcal{O}_N^{(k)}(z_{21}^u) \\
& - \mathbf{I}_{\lambda_N+k-1}\Big(\sqrt{-u\bar{u}z_{12}^2\partial^2}\Big)\mathcal{O}_N^{\mu,(k)}(z_{21}^u)\Big) + (z_1\leftrightarrow z_2,\mu\leftrightarrow\nu)\Bigg]\Bigg\}, \qquad (3.144)
\end{aligned}
$$

where $C_k^{(a)}(z)$ are Gegenbauer's polynomials (related to Jacobi's), the differential operator $\mathbf{D}_N^{\mu\nu}$ is defined as

$$
\begin{aligned}
\mathbf{D}_N^{\mu\nu} = & \,\mathbb{C}_0^N\frac{\mathcal{I}^{\mu\nu}(z_{12})}{z_{12}^2} + \mathbb{C}_1^N\partial_1^\mu\partial_2^\nu + \mathbb{C}_2^N\mathcal{I}_\rho^\mu(z_{12})\mathcal{I}_\sigma^\nu(z_{12})\partial_2^\rho\partial_1^\sigma \\
& + (\widetilde{\ell}_N-N)\mathbb{C}_3^N\Big(\partial_1^\mu z_{21}^\nu+\partial_2^\nu z_{12}^\mu\Big)\frac{1}{z_{12}^2}\,. \qquad (3.145)
\end{aligned}
$$

and the following notation was introduced:

$$
\tau_N = \frac{1}{2}t_N\,,\quad \lambda_N = \ell_N - \frac{D}{2}\,,\quad j_N = \frac{\ell_N+N}{2}\,. \qquad (3.146)
$$

The *descendants* of the operator $\mathcal{O}_{\alpha_1\cdots\alpha_N}(y)$ are given by

$$
\begin{aligned}
\mathcal{O}_N^{(k)}(y) &= \partial_y^{\alpha_1}\cdots\partial_y^{\alpha_k}\mathcal{O}_{\alpha_1\cdots\alpha_k\alpha_{k+1}\cdots\alpha_N}(y)z_{12}^{\alpha_{k+1}}\cdots z_{12}^{\alpha_N}\,, \\
\mathcal{O}_N^{\alpha,(k)}(y) &= \partial_y^{\alpha_1}\cdots\partial_y^{\alpha_k}\,y^\alpha\,\mathcal{O}_{\alpha_1\cdots\alpha_k\alpha_{k+1}\ldots\alpha_N}(y)z_{12}^{\alpha_{k+1}}\cdots z_{12}^{\alpha_N}\,, \qquad (3.147)
\end{aligned}
$$

and in Eq. (3.144), the derivatives ∂^2 of the expansion of $\mathbf{I}_\nu$ must be understood as

$$
\partial^2\mathcal{O}_N^{(k)}(z_{21}^u) = \partial_y^2\mathcal{O}_N^{(k)}(z_{21}^u+y)\Big|_{y=0}\,. \qquad (3.148)
$$

Beyond LO, the COPE in Eq. (3.144) is valid at the Fisher-Wilson fixed point [97] of the $\beta-$function, which happens for a particular spacetime dimension $D = 4-2\epsilon$ with $\epsilon \neq 0$. Nevertheless, one can make use of this form of the COPE beyond LO for the four-dimensional QCD by combining it with the RG equations, as discussed in Sect. 3.2.

The two coefficients c_1^N and c_2^N of Eq. (3.134) which are hidden inside $\mathbb{C}_0,\ldots,\mathbb{C}_3$ in operator $\mathbf{D}_N^{\mu\nu}$ (3.145) are the last unknown pieces to have a complete OPE. By taking the forward limit of the OPE (3.144), this is to consider $\langle N(p)|\mathbb{T}\{j_\mu(z_1)j_\nu(z_2)\}|N(p)\rangle$, they can be related to the two structure functions of DIS. For details, vid. Refs. [47, 111].

3.4.1 Born approximation and light-ray representation

As explained in Sect. 3.2, at LO in the strong coupling constant the four-dimensional QCD is an exact conformal field theory. In this case, the coefficients c_1^N and c_2^N can be evaluated for null anomalous dimensions and we can drop terms regular in z_{12}^2 as they produce Dirac-deltas and derivatives of those (after Fourier transform to momentum space). Consequently, they do not take part of observables. In Minkowski spacetime, the result is:

$$
\begin{aligned}
&\mathbb{T}\{j^\mu(z_1)j^\nu(z_2)\} = \\
&= \frac{1}{i\pi^2}\sum_{N>0,\,\text{even}}\frac{\rho_N}{N+1}\int_0^1 du\,(u\bar{u})^N\Bigg\{\frac{1}{(-z_{12}^2+i0)^2}\Bigg[(N+1)g_{\mu\nu}\left(1-\frac{1}{4}\frac{u\bar{u}}{N+1}z_{12}^2\partial^2\right) \\
&\quad+\frac{1}{2N}z_{12}^2(\partial_1^\mu\partial_2^\nu-\partial_1^\nu\partial_2^\mu)+\left(1-\frac{1}{4}\frac{u\bar{u}}{N}z_{12}^2\partial^2\right)\left(\frac{\bar{u}}{u}z_{21}^\mu\partial_1^\nu+\frac{u}{\bar{u}}z_{12}^\nu\partial_2^\mu\right) \\
&\quad-\frac{1}{4}\frac{u\bar{u}}{N(N+1)}z_{12}^2\partial^2\left(z_{21}^\nu\partial_1^\mu+z_{12}^\mu\partial_2^\nu\right)-\frac{z_{12}^\mu z_{12}^\nu}{N+1}u\bar{u}\partial^2\left(1-\frac{1}{4}\frac{u\bar{u}}{N+2}z_{12}^2\partial^2\right)\Bigg]\mathcal{O}_N^{(0)}(z_{21}^u) \\
&\quad-\frac{1}{(-z_{12}^2+i0)}\Bigg[-\frac{1}{4}N(\bar{u}-u)\,g_{\mu\nu}-\frac{\bar{u}-u}{4(N+1)}(z_{21}^\nu\partial_1^\mu+z_{12}^\mu\partial_2^\nu) \\
&\qquad+\frac{1}{2}\left(\bar{u}\,z_{21}^\mu\partial_1^\nu-u\,z_{12}^\nu\partial_2^\mu\right)+\frac{N}{2(N+2)(N-1)}\left(z_{21}^\nu\partial_1^\mu-z_{12}^\mu\partial_2^\nu\right) \\
&\qquad+\frac{1}{4}\frac{N(N^2+N+2)}{(N+1)(N+2)(N-1)}\left(\frac{u}{\bar{u}}z_{12}^\nu\partial_2^\mu-\frac{\bar{u}}{u}z_{21}^\mu\partial_1^\nu\right) \\
&\qquad+\frac{z_{12}^\mu z_{12}^\nu}{(-z_{12}^2+i0)}(\bar{u}-u)\frac{N}{N+1}\left(1-\frac{1}{2}\frac{u\bar{u}}{N+2}z_{12}^2\partial^2\right)\Bigg]\mathcal{O}_N^{(1)}(z_{21}^u) \\
&\quad-\frac{z_{12}^\mu z_{12}^\nu}{(-z_{12}^2+i0)}\left[\frac{N^2+N+2}{4(N+1)(N+2)}-u\bar{u}\frac{N(N-1)}{(N+1)(N+2)}\right]\mathcal{O}_N^{(2)}(z_{21}^u)\Bigg\}+\cdots,
\end{aligned}
\tag{3.149}
$$

The ellipsis represents the regular terms in z_{12}^2 and

$$
\rho_N = i^{N-1}\frac{(2N+1)!}{(N-1)!N!N!}\,. \tag{3.150}
$$

The multiplicatively renormalizable (conformal) leading-twist operators that enter the COPE are given by

$$
n_{\alpha_1}\cdots n_{\alpha_N}\mathcal{O}_N^{\alpha_1\cdots\alpha_N}(y) = \frac{\Gamma(3/2)\Gamma(N)}{\Gamma(N+1/2)}\left(\frac{i\partial^+}{4}\right)^{N-1}\bar{q}(y)\gamma^+ C_{N-1}^{3/2}\left(\frac{\overrightarrow{D}^+-\overleftarrow{D}^+}{\overrightarrow{D}^++\overleftarrow{D}^+}\right)q(y)\,. \tag{3.151}
$$

Here, $\gamma^+ = n\gamma = \not{n}$, $\partial^+ = n\partial$ and $D^+ = nD$ where D^α is the QCD covariant derivative. At LO, $D^\alpha \to \partial^\alpha$. The arrows on top of the derivatives indicate the quark field they are acting on. See Ref. [95] for a proof of Eq. (3.151).

Thus far, the chosen basis for expansion was that of vector operators. To the product of two spin-1 currents, there is also a contribution coming from the basis of axial-vector operators $\mathcal{O}^{A,\alpha_1\cdots\alpha_N}$ which will account for the contribution of axial GPDs. In Ch. 4 we deal with scalar and pseudo-scalar targets at LO, so the vector component to the OPE, this is Eq. (3.149), is enough for our purposes. For a discussion on the axial component, see [47].

Although the descendants $\mathcal{O}_N^{(k)}$ consist of derivatives on leading-twist conformal operators $\mathcal{O}_{\alpha_1\cdots\alpha_N}$, they are not twist-2 operators themselves as not all traces have been removed. For example, we could considering the expansion of $\mathcal{O}_N^{(0)}$ around $y = (z_1+z_2)/2$. One obtains local operators of the form:

$$z_{12}^{\beta_1}\cdots z_{12}^{\beta_k}\, z_{12}^{\alpha_1}\cdots z_{12}^{\alpha_N}\partial_{\beta_1}\cdots\partial_{\beta_k}\mathcal{O}_{\alpha_1\cdots\alpha_N}\left(\frac{z_1+z_2}{2}\right), \tag{3.152}$$

for which the removal of traces in indices β_i has not been taken care of yet. Leading-twist operators are easy to relate to GPDs, so we would like to write the operators $\mathcal{O}_N^{(0)}$, $\mathcal{O}_N^{(1)}$ and $\mathcal{O}_N^{(2)}$ that appear in the OPE (3.149) by means of their LT components. In Eq. (D.19) of App. D, we give the projection of a scalar operator onto its LT component:

$$\begin{aligned}
[\mathcal{O}(z)]_{\rm LT} &= \mathcal{O}(z) + \sum_{k=1}^{\infty}\int_0^1 dw\,\left(\frac{-z^2}{4}\right)^k \frac{(\partial^2)^k}{k!(k-1)!}\frac{\bar{w}^{k-1}}{w^k}\mathcal{O}(wz) \\
&= \mathcal{O}(z) - \frac{1}{4}z^2\int_0^1 \frac{dw}{w}\partial^2\mathcal{O}(wz) + \frac{1}{32}z^4\int_0^1 \frac{dw}{w}\frac{\bar{w}}{w}\partial^4\mathcal{O}(wz) + O(z^6)\,,
\end{aligned} \tag{3.153}$$

where

$$\partial^2 = \frac{\partial}{\partial z^\mu}\frac{\partial}{\partial z_\mu} \quad \text{and} \quad \bar{w} = 1-w\,. \tag{3.154}$$

Inverting the above relation:

$$\begin{aligned}
\mathcal{O}(z) &= [\mathcal{O}(z)]_{\rm LT} + \frac{1}{4}z^2\int_0^1 \frac{dw}{w}\partial^2\mathcal{O}(wz) - \frac{1}{32}z^4\int_0^1 \frac{dw}{w}\frac{\bar{w}}{w}\partial^4\mathcal{O}(wz) + O(z^6) \\
&= [\mathcal{O}(z)]_{\rm LT} + \frac{1}{4}z^2\int_0^1 \frac{dw}{w}\left[\partial^2\mathcal{O}(wz)\right]_{\rm LT} + \frac{1}{32}z^4\int_0^1 dw\,\frac{\bar{w}}{w^3}\left[\partial^4\,\mathcal{O}(wz)\right]_{\rm LT} + O(z^6)\,.
\end{aligned} \tag{3.155}$$

Now, we can make use of this expression to obtain the corresponding one for $\mathcal{O}_N^{(0)}$, $\mathcal{O}_N^{(1)}$ and $\mathcal{O}_N^{(2)}$ to the required accuracy. Inspection of Eq. (3.149) reveals that $\mathcal{O}_N^{(0)}(z_{21}^u)$ should be kept up to $O(z_{12}^4)$ as higher orders would produce polynomials in z_{12}^2. These polynomials give rise to Dirac deltas and derivatives of those which do not take part of observables. Likewise, operator $\mathcal{O}_N^{(1)}(z_{21}^u)$ should be kept up to $O(z_{12}^2)$, while operator $\mathcal{O}_N^{(2)}(z_{21}^u)$ up to $O(z_{12}^0)$. Considering $z_1 = z$, $z_2 = 0$ so that $z_{21}^u = uz$, one finds:

$$\begin{aligned}
\mathcal{O}_N^{(0)}(uz) &= \left[\mathcal{O}_N^{(0)}(uz)\right]_{\rm LT} - \frac{z^2}{4}\int_0^1 \frac{dw}{w}w^N\left[u^2w^2t\left[\mathcal{O}_N^{(0)}(uwz)\right]_{\rm LT} - 2uwN\left[\mathcal{O}_N^{(1)}(uwz)\right]_{\rm LT}\right] \\
&\quad + \frac{z^4}{32}\int_0^1 dw\,\frac{\bar{w}}{w^3}w^N\Big\{u^4w^4t^2\left[\mathcal{O}_N^{(0)}(uwz)\right]_{\rm LT} - 4Nu^3w^3t\left[\mathcal{O}_N^{(1)}(uwz)\right]_{\rm LT} \\
&\quad + 4N(N-1)u^2w^2\left[\mathcal{O}_N^{(2)}(uwz)\right]_{\rm LT}\Big\} + O(z^6)\,,
\end{aligned} \tag{3.156}$$

$$\mathcal{O}_N^{(1)}(uz) = \Big[\mathcal{O}_N^{(1)}(uz)\Big]_{\rm LT} - \frac{z^2}{4}\int_0^1 \frac{dw}{w} w^{N-1}\Big[u^2 w^2 t\Big[\mathcal{O}_N^{(1)}(uwz)\Big]_{\rm LT} - 2uw(N-1)\Big[\mathcal{O}_N^{(2)}(uwz)\Big]_{\rm LT}\Big] + O(z^4)\,, \tag{3.157}$$

$$\mathcal{O}_N^{(2)}(uz) = [\mathcal{O}_N^2(uz)]_{\rm LT} + O(z^2)\,. \tag{3.158}$$

Here, $t = \Delta^2 = (p' - p)^2$ is the usual Madelstam variable and all above expressions are considered as matrix elements, i.e. $\langle p'|\mathcal{O}_N^{(k)}|p\rangle$.

The next step is to connect these leading-twist operators to the non-local light-ray operators defined as:

$$\mathscr{O}(\lambda_1, \lambda_2) = \sum_f \left(\frac{e_f}{e}\right)^2 \frac{1}{2}\left[\bar{\mathfrak{q}}_f(\lambda_1 z)\not{z}\mathfrak{q}_f(\lambda_2 z) - (\lambda_1 \leftrightarrow \lambda_2)\right]_{\rm LT}, \tag{3.159}$$

where

$$\lambda_{12} = \frac{z_{12} n'}{nn'}\,, \quad z_i = \lambda_i z\,, \quad \lambda_i \in \mathbb{R}\,, \tag{3.160}$$

and f stands for the flavor index of the quark field $\mathfrak{q}$. Likewise, e_f is the electric charge of the corresponding quark. The light-ray operators are directly connected to GPDs and can be written by means of *double distributions* (DDs) in a straightforward way. We will prove this in Sect. 3.4.2.

For example, $\mathcal{O}_N^{(0)}$ is related to $\mathscr{O}$ by the integral:

$$\mathscr{O}(\lambda_1, \lambda_2) = \sum_{\substack{N>0,\\ \text{even}}} \rho_N \lambda_{12}^{N-1} \int_0^1 du\,(u\bar{u})^N \Big[\mathcal{O}_N^{(0)}(\lambda_{21}^u z)\Big]_{\rm LT}. \tag{3.161}$$

The coefficient ρ_N was introduced earlier in Eq. (3.150). Similar formulas can be found for $\mathcal{O}_N^{(1)}$ and $\mathcal{O}_N^{(2)}$, see Ref. [47]. The idea now is to make use of these integral relations to write the contributions to the OPE (3.149), which has the generic form

$$\sum_{\substack{N>0,\\ \text{even}}} \rho_N f(N) \int_0^1 du\,(u\bar{u})^N g(u) \Big[\mathcal{O}_N^{(k)}(uz)\Big]_{\rm LT}, \tag{3.162}$$

by means of the light-ray operators $\mathscr{O}$.

Finally, the result for the vector contribution to the OPE of the Compton tensor at LO is:

$$\begin{aligned}
T^{\mu\nu} &= i\int d^4z\, e^{iq'z}\langle p'|\mathbb{T}\{j^\mu(z)j^\nu(0)\}|p\rangle \\
&= \frac{1}{i\pi^2} i\int d^4z\, e^{iq'z} \\
&\times\Bigg\{\frac{1}{(-z^2+i0)^2}\Big[g^{\mu\nu}\mathscr{O}(1,0) - z^\mu\partial^\nu\int_0^1 du\,\mathscr{O}(\bar{u},0) - z^\nu(\partial^\mu - i\Delta^\mu)\int_0^1 dv\,\mathscr{O}(1,v)\Big] \\
&- \frac{1}{-z^2+i0}\Big[\frac{i}{2}(\Delta^\nu\partial^\mu - \Delta^\mu\partial^\nu)\int_0^1 du\int_0^{\bar{u}} dv\,\mathscr{O}(\bar{u},v) - \frac{t}{4}z^\mu\partial^\nu\int_0^1 du\,u\int_0^{\bar{u}} dv\,\mathscr{O}(\bar{u},v)\Big] \\
&+ \frac{t}{2}\frac{z^\mu z^\nu}{(-z^2+i0)^2}\int_0^1 du\,\bar{u}\int_0^{\bar{u}} dv\,\mathscr{O}(\bar{u},v)
\end{aligned}$$

$$+ \frac{g^{\mu\nu}}{4(-z^2+i0)} \left[-\int_0^1 du \int_0^{\bar{u}} dv\, \mathcal{O}_1(\bar{u},v) + \int_0^1 dv\, \mathcal{O}_2(1,v) \right]$$
$$+ \frac{1}{4(-z^2+i0)} (z^\mu \partial^\nu + z^\nu \partial^\mu - iz^\mu \Delta^\nu) \int_0^1 du \int_0^{\bar{u}} dv \left((\ln \bar{\tau}) \mathcal{O}_1(\bar{u},v) + \frac{v}{\bar{v}} \mathcal{O}_2(\bar{u},v) \right)$$
$$+ \frac{1}{2(-z^2+i0)} (-z^\mu \partial^\nu + z^\nu \partial^\mu + iz^\mu \Delta^\nu) \int_0^1 du \int_0^{\bar{u}} dv\, \frac{\tau}{\bar{\tau}} \left(-\mathcal{O}_1(\bar{u},v) + \frac{\bar{u}}{u} \mathcal{O}_2(\bar{u},v) \right)$$
$$+ \frac{1}{4(-z^2+i0)} z^\nu (\partial^\mu - i\Delta^\mu) \int_0^1 du \int_0^{\bar{u}} dv\, \frac{v}{\bar{v}} \left[-2\left(1 + \frac{2\tau}{\bar{\tau}}\right) \mathcal{O}_1(\bar{u},v) + \frac{v}{\bar{v}} \mathcal{O}_2(\bar{u},v) \right]$$
$$+ \frac{1}{4(-z^2+i0)} z^\nu (\partial^\mu - i\Delta^\mu) \int_0^1 dv\, \frac{v}{\bar{v}} \mathcal{O}_2(1,v)$$
$$+ \frac{1}{2(-z^2+i0)} z^\mu \partial^\nu \int_0^1 du \int_0^{\bar{u}} dv\, (\ln \bar{u} + u) \mathcal{O}_1(\bar{u},v) + \bar{u} \mathcal{O}_2(\bar{u},v)$$
$$- \frac{1}{4(-z^2+i0)} z^\mu \partial^\nu \int_0^1 du \int_0^{\bar{u}} dv \left(1 + \frac{4\tau}{\bar{\tau}}\right) \mathcal{O}_2(\bar{u},v)$$
$$- \frac{z^\mu z^\nu}{(-z^2+i0)^2} \int_0^1 du \int_0^{\bar{u}} dv \left[(\ln \bar{\tau} + \ln \bar{u} + u) \mathcal{O}_1(\bar{u},v) + \left(\frac{v}{\bar{v}} + \bar{u}\right) \mathcal{O}_2(\bar{u},v) \right]$$
$$+ \frac{z^\mu z^\nu}{4(-z^2+i0)} \left[i\Delta\partial + \frac{t}{2} \right] \int_0^1 du \int_0^{\bar{u}} dv\, \frac{v}{\bar{v}} \left(\frac{2}{\bar{\tau}} - 1 \right) \mathcal{O}_1(\bar{u},v)$$
$$- \frac{z^\mu z^\nu}{2(-z^2+i0)} \left[i\Delta\partial + \frac{t}{4} \right] \int_0^1 du \int_0^{\bar{u}} dv \left(\ln \bar{\tau} + \frac{2\tau}{\bar{\tau}} \right) \mathcal{O}_1(\bar{u},v) \Bigg\}, \tag{3.163}$$

where

$$\tau = \frac{uv}{\bar{u}\bar{v}} \quad \text{and} \quad \bar{\tau} = 1 - \tau, \tag{3.164}$$

while the operators $\mathcal{O}$, $\mathcal{O}_1$ and $\mathcal{O}_2$ must be understood as matrix elements $\langle p'|\mathcal{O}|p\rangle$, $\langle p'|\mathcal{O}_1|p\rangle$, $\langle p'|\mathcal{O}_2|p\rangle$, respectively, defining

$$\mathcal{O}_1(\lambda_1,\lambda_2) = (i\Delta\partial_z)\mathcal{O}(\lambda_1,\lambda_2), \tag{3.165}$$
$$\mathcal{O}_2(\lambda_1,\lambda_2) = \mathcal{O}_1(\lambda_1,\lambda_2) + \frac{t}{2}\mathcal{O}(\lambda_1,\lambda_2). \tag{3.166}$$

3.4.2 Light-ray representation and GPDs

For GPDs, we follow the convention of Ref. [69] which is the one used in Ch. 2. Thus, for a (pseudo-)scalar hadron we have the GPD given by the following Fourier transform:

$$H_f(x,\xi,t) = \int \frac{d\lambda}{2\pi} e^{ix(\bar{p}z)} \langle p'|\bar{\mathfrak{q}}_f(-z/2)\not{n}\mathfrak{q}_f(z/2)|p\rangle\big|_{z=\lambda n}, \quad \lambda \in \mathbb{R}. \tag{3.167}$$

Spacetime translation provides,

$$\mathfrak{q}_f(\lambda n/2) = e^{i(\lambda n/2 - z_1)P} \mathfrak{q}_f(z_1) e^{-i(\lambda n/2 - z_1)P}, \tag{3.168}$$

where P is the momentum operator. Consequently,

$$\langle p'|\bar{\mathfrak{q}}_f(-z/2)\not{n}\mathfrak{q}_f(z/2)|p\rangle\big|_{z=\lambda n} = e^{-i\Delta z_1} e^{-i\xi(\bar{p}n)\lambda} \langle p'|\bar{\mathfrak{q}}_f(z_2)\not{n}\mathfrak{q}_f(z_1)|p\rangle, \tag{3.169}$$

where it was used that $-\Delta n = \xi 2\bar{p}n$ and $z_2 = -\lambda n + z_1$.

Introducing the last expression into Eq. (3.167) renders

$$H_f(x,\xi,t) = \int \frac{d\lambda}{2\pi}\, e^{i\lambda(\bar{p}n)(x-\xi)-i\Delta z_1}\langle p'|\bar{\mathfrak{q}}_f(z_2)\not{n}\mathfrak{q}_f(z_1)|p\rangle\,, \quad z_1 - z_2 = \lambda n\,. \tag{3.170}$$

Let us extract the quark correlator by means of the Fourier transform on the variable x. Defining $\Lambda = \lambda(\bar{p}n)$ and $\Lambda_{12} = \lambda_{12}(\bar{p}n)$, we have:

$$\begin{aligned}\int_{-1}^{1} dx\, e^{-ix\Lambda_{12}} H_f(x,\xi,t) &= \int_{-1}^{1} dx\, e^{-ix\Lambda_{12}} \frac{1}{\bar{p}n} \int \frac{d\Lambda}{2\pi}\, e^{i\Lambda(x-\xi)-i\Delta z_1}\langle p'|\bar{\mathfrak{q}}_f(z_2)\not{n}\mathfrak{q}_f(z_1)|p\rangle \\ &= \frac{1}{\bar{p}n} e^{-i\lambda_{12}(\bar{p}n)\xi - i\Delta z_1}\langle p'|\bar{\mathfrak{q}}_f(z_2)\not{n}\mathfrak{q}_f(z_1)|p\rangle\,,\end{aligned} \tag{3.171}$$

where $z_1 - z_2 = \lambda_{12} n$. Choosing $\lambda_{12} = \lambda_1 - \lambda_2$ so that $z_i = \lambda_i n$ ($i \in \{1,2\}$), we get:

$$\Delta z_1 = -2\xi(\bar{p}n)\lambda_1 \tag{3.172}$$

which allows us to write the correlator as

$$\langle p'|\bar{\mathfrak{q}}_f(z_2)\not{n}\mathfrak{q}_f(z_1)|p\rangle = (\bar{p}n)\int_{-1}^{1} dx\, e^{-i(\bar{p}n)[\lambda_1(\xi+x)+\lambda_2(\xi-x)]} H_f(x,\xi,t)\,. \tag{3.173}$$

From this formula one notices that the exchange $z_1 \leftrightarrow z_2$ (or, similarly, $\lambda_1 \leftrightarrow \lambda_2$) is equivalent to the switch $x \to -x$. Therefore, we may consider the following non-local operator evaluated at light-cone positions:

$$\mathcal{O}_f(\lambda_2,\lambda_1) = \frac{1}{2}\bar{\mathfrak{q}}_f(\lambda_2 n)\not{n}\mathfrak{q}_f(\lambda_1 n) - (\lambda_2 \leftrightarrow \lambda_1)\,, \quad z_i = \lambda_i n\,, \tag{3.174}$$

assuming Wilson lines implicitly and $n^2 = 0$. Its expectation value with states of momentum p and $p' \neq p$ is therefore,

$$\begin{aligned}\langle p'|\mathcal{O}_f(\lambda_2,\lambda_1)|p\rangle = \frac{\bar{p}n}{2}\Bigg[&\int_{-1}^{1} dx\, e^{-i(\bar{p}n)[\lambda_1(\xi+x)+\lambda_2(\xi-x)]} H_f(x,\xi,t) \\ &- \int_{-1}^{1} d(-y)\, e^{-i(\bar{p}n)[\lambda_1(\xi+y)+\lambda_2(\xi-y)]} H_f(-y,\xi,t)\Bigg]\,,\end{aligned} \tag{3.175}$$

where in the second line we renamed $x \to -y$. With $\int_{-1}^{1} d(-y) = -\int_{1}^{-1} dy \xrightarrow{y\to x} \int_{-1}^{1} dx$, we finally obtain

$$\langle p'|\mathcal{O}_f(\lambda_2,\lambda_1)|p\rangle = 2(\bar{p}n)\int_{-1}^{1} dx\, e^{-i(\bar{p}n)[\lambda_1(\xi+x)+\lambda_2(\xi-x)]}\frac{H_f^{(+)}(x,\xi,t)}{4}\,. \tag{3.176}$$

Comparison with Eq. (3.21) from [47], the GPD $H_q(x,\xi,t)$ used there (with flavor index q) is indeed a quarter of the usual C-even part of the GPD denoted here as $H_f^{(+)}(x,\xi,t) = H_f(x,\xi,t) - H_f(-x,\xi,t)$ for flavor index f.

According to [47] and [112], for the isoscalar[7] hadron one may consider the following *double distribution* (DD) representation:

$$\langle p'|\mathcal{O}_f(\lambda_1,\lambda_2)|p\rangle = i\iint_{\mathbb{D}} d\beta d\alpha\, e^{-i\ell_{\lambda_1,\lambda_2}n}\left[2(\bar{p}n)h_f(\beta,\alpha,t)-(\Delta n)g_f(\beta,\alpha,t)\right]\,, \tag{3.177}$$

where h_f, g_f are the so-called double distributions for quark flavor f, the vector $\ell_{\lambda_1,\lambda_2}$ is

$$\ell_{\lambda_1,\lambda_2} = -\lambda_1\Delta - \lambda_{12}\left[\beta\bar{p}-\frac{1}{2}(\alpha+1)\Delta\right]\,, \tag{3.178}$$

and the domain $\mathbb{D}$ for integration with respect to variables α and β has been defined as

$$\iint_{\mathbb{D}} d\beta d\alpha \;=\; \int_{-1}^{1} d\beta\int_{|\beta|-1}^{1-|\beta|} d\alpha\,. \tag{3.179}$$

Function g_f is related to the D-term as defined in [112] via

$$D(\beta,t) = -i\sum_f\int_{|\beta|-1}^{1-|\beta|} d\alpha\, e^{-i\ell_{\lambda_1,\lambda_2}n}g_f(\beta,\alpha,t)\,. \tag{3.180}$$

Taking the Fourier transform to isolate the GPD, one finds

$$\frac{1}{4}H_f^{(+)}(x,\xi,t) = \iint_{\mathbb{D}} d\beta d\alpha\,\delta(x-\beta-\alpha\xi)\left[h_f(\beta,\alpha,t)+\xi g_f(\beta,\alpha,t)\right]\,. \tag{3.181}$$

Nevertheless, Eq. (3.177) can be further simplified if one assumes h_f, g_f to vanish in the boundaries of the domain $\mathbb{D}$. Expressing $\bar{p}n$ and Δn by derivatives with respect to β and α, respectively, acting on the exponential of Eq. (3.177), one gets via integration by parts:

$$\langle p'|\mathcal{O}_f(\lambda_1,\lambda_2)|p\rangle = \frac{2i}{\lambda_{12}}\iint_{\mathbb{D}} d\beta d\alpha\, e^{-i\ell_{\lambda_1,\lambda_2}n}\Phi_f^{(+)}(\beta,\alpha,t)\,,\quad z_i=\lambda_i n\,, \tag{3.182}$$

where

$$\Phi_f^{(+)}(\beta,\alpha,t) = \partial_\beta h_f + \partial_\alpha g_f\,. \tag{3.183}$$

The function $\Phi^{(+)}$ satisfies to be even under $(\beta,\alpha)\to(-\beta,-\alpha)$:

$$\Phi_f^{(+)}(-\beta,-\alpha,t) = \Phi_f^{(+)}(\beta,\alpha,t) \tag{3.184}$$

and consequently,

$$\iint_{\mathbb{D}} d\beta d\alpha\,\alpha\Phi_f^{(+)}(\beta,\alpha,t)=0\,,\qquad \iint_{\mathbb{D}} d\beta d\alpha\,\beta\Phi_f^{(+)}(\beta,\alpha,t)=0\,. \tag{3.185}$$

Because h_f and g_f vanish in the frontier of $\mathbb{D}$, it follows:

$$\iint_{\mathbb{D}} d\beta d\alpha\,\Phi_f^{(+)}(\beta,\alpha,t)=0\,. \tag{3.186}$$

[7] Processes such as DVCS, TCS and DDVCS project the C-even part of GPDs (this is $H_f^{(+)}$). Quark combinations of this kind have isospin $I=0$ so they are isoscalars.

If in Eq. (3.174) we change $n \to z$ with $z^2 \neq 0$, such that $z_{12} = z_1 - z_2 = \lambda_{12} z$, we have the alternative operator

$$\mathscr{O}_f(\lambda_1, \lambda_2) = \frac{1}{2}\left[\bar{\mathrm{q}}_f(\lambda_1 z) \not{z}\, \mathrm{q}_f(\lambda_2 z) - (\lambda_1 \leftrightarrow \lambda_2)\right]_{\mathrm{LT}}, \quad \lambda_{12} = \frac{z_{12} n'}{z n'}, \quad z_i = \lambda_i z\,. \tag{3.187}$$

This one is the light-ray operator, up to quark charges, previously introduced in Eq. (3.159). Since $\mathscr{O}_f$ has an expression similar to that of the operator $\mathcal{O}_f$ (3.174) up to $n \to z$ and the correlator of the latter has its dependence on the spacetime position isolated on an exponential, we might as well write:

$$\langle p' | \mathscr{O}_f(\lambda_1, \lambda_2) | p \rangle = \frac{2i}{\lambda_{12}} \iint_{\mathbb{D}} d\beta d\alpha \left[e^{-i \ell_{\lambda_1, \lambda_2} z}\right]_{\mathrm{LT}} \Phi_f^{(+)}(\beta, \alpha, t)\,. \tag{3.188}$$

Now it is manifest that the operators that partake of the OPE in Eq. (3.149) can be expressed by means of DDs ($\Phi^{(+)}$) and, therefore, by GPDs ($H^{(+)}$), *quod erat demonstrandum*. This is a consequence of our starting point: symmetric and traceless (LT) conformal operators.

Since $z^2 \neq 0$ in (3.187), taking its LT component makes the deviation from the light-cone to be given by the traces that are proportional to $z^2 \neq 0$ powers. After Fourier transform, they render $1/\mathbb{Q}^2$ powers (with $\mathbb{Q}$ the scale of the process), i.e. the kinematic-twist corrections. **In other words, deviation from the light-cone in operators translates to a relaxation in the Björken limit for observables.**

When summing over flavors, we will use the notation:

$$\Phi^{(+)}(\beta, \alpha, t) = \sum_f \left(\frac{e_f}{e}\right)^2 \Phi_f^{(+)}(\beta, \alpha, t) \tag{3.189}$$

and

$$H^{(+)}(x, \xi, t) = \sum_f \left(\frac{e_f}{e}\right)^2 H_f^{(+)}(x, \xi, t)\,. \tag{3.190}$$

The function $\Phi^{(+)}$ can be related to the GPD $H^{(+)}$ by introducing the identity

$$1 = \int_{-1}^{1} dx\, \delta(x - \beta - \alpha\xi) \tag{3.191}$$

in Eq. (3.182):

$$\frac{1}{4} \partial_x H^{(+)}(x, \xi, t) = \iint_{\mathbb{D}} d\beta d\alpha\, \delta(x - \beta - \alpha\xi) \Phi^{(+)}(\beta, \alpha, t)\,. \tag{3.192}$$

To summarize, the main results of this chapter correspond to Eqs. (3.163), (3.187), (3.188) and (3.192). Making use of the matrix elements of light-ray operators (3.187, 3.188), we can express the vector part of the Compton tensor OPE (3.163) by means of the DD $\Phi^{(+)}$ (3.183). In turn, thanks to Eq. (3.192), we can transit from the DD representation to the GPD one. These formulas give us the necessary tools to compute the kinematic higher-twist corrections of DDVCS off the pseudo-scalar and scalar targets in chapter 4.

4

DDVCS off a (pseudo-)scalar target

In this chapter, we consider the electroproduction of a muon pair on a spin-0 target h,

$$e^- + h \to e^- + h' + \mu^+ + \mu^- \, , \tag{4.1}$$

for which DDVCS represents the Compton scattering component of the amplitude of the process. In (4.1), h' represents the hadron h after being struck by the electron beam.

The goal of this chapter is to provide the theoretical and phenomenological description of DDVCS at LO including the kinematic twist-3 and twist-4 corrections. This is to consider kinematic effects entering the amplitude with at most a dumping factor of the form $|t|/\mathbb{Q}^2$ and $M^2/\mathbb{Q}^2$, with $\mathbb{Q}^2$ the energy scale of the process. Our interest in these corrections comes from their magnitude and their impact on the 3D imaging of the hadron. In facilities such as JLab, they may be typically of the order of 10% to 30%, therefore they are large enough to be measurable and to impact the GPD extraction from data. Also, the 3D imaging given by the tomography formula (1.85) requires a Fourier transform on the transverse component of the momentum transfer, hence a correct description of data corresponding to a non-negligible value of the Mandelstam variable t, with respect to the scale $\mathbb{Q}^2$, is required.

As DDVCS serves as a single framework to study DVCS and TCS as well, we provide results for the three processes.

4.1 Description of the problem

In chapter 2 we discussed DDVCS on a nucleon target (spin 1/2). The study was done at leading order (LO) in the QCD coupling constant and leading twist (LT), this is neglecting corrections inversely proportional to the energy scale of the process. In order words, we considered $|t|/\mathbb{Q}^2 \to 0$ and $M^2/\mathbb{Q}^2 \to 0$, where t is the Mandelstam variable, M is the mass of the target and $\mathbb{Q}^2 = Q^2 + Q'^2$ is the scale of DDVCS given by the sum of the virtualities of the two photons involved (using the notation of Ch. 2). These finite-t and target-mass corrections are referred to as kinematic power corrections. This accuracy (LO + LT) was sufficient to assess the prospects of measuring DDVCS at current and future facilities such as JLab and the EIC.

The promising results shown in Sect. 2.6 motivates a higher-precision study. In this regard, we are interested in the aforementioned kinematic corrections because, for instance, in experiments such as JLab they may be typically of the order of 10% to 30%. In DVCS analysis such as [44], two cuts on experimental data are imposed: 1) $\mathbb{Q}^2 = Q^2 > 1.5\ \text{GeV}^2$ and 2) $|t|/\mathbb{Q}^2 < 0.2$, which authors claim to be enough to fulfill the requirements of collinear factorization and leading twist dominance. Including the corrections on t allows us to increase the range of useful experimental data. This is positive not only for a highly precise GPD extraction, but it is also fundamental for the study of hadron tomography. Tomography consists of a Fourier transform in the momentum transfer to the target (vid. Eq. (1.85)) and, as a consequence, data on a large range of $|t|$ is needed.

In this chapter, we focus on the calculation of kinematic power corrections to DDVCS off a spin-0 target h through the electroproduction of a muon-antimuon pair:

$$h(p) + e^-(k) \to h(p') + e^-(k') + \mu^-(\ell_-) + \mu^+(\ell_+)\,, \tag{4.2}$$

where the photon virtualities are $-q^2 = -(k-k')^2 = Q^2 > 0$ and $q'^2 = (\ell_- + \ell_+)^2 = Q'^2 > 0$. In turn, the Mandelstam variable is given by the difference between the momenta of the initial- and final-state proton: $t = \Delta^2 = (p' - p)^2 < 0$.

We are interested in the spin-0 target because it is the simplest case, hence the number of CFFs is the minimal possible: one at LT and five when kinematic power corrections are taken into account. Consequently, the number of experimental measurements required to extract the different CFFs and constrain the GPD is less than next hadron in spin, for example a nucleon. A spin-1/2 hadron is described by four CFFs at LT and eighteen beyond [113].

The importance of the spin-0 target is well acknowledged by the scientific community: cf. [114] for a modeling of the ^{4}He GPD and Refs. [115–117] for the modeling of the pion GPD, mainly accessible through the Sullivan process [118]. This process consists of a DVCS off the proton's virtual pion cloud. Moreover, kinematic higher-twist corrections have already been computed for scalar and pseudo-scalar meson-antimeson and meson-antimeson-photon production, see Refs. [119, 120]. The latter processes are factorized by means of *generalized distribution amplitudes* (GDAs) which are closely related to GPDs as they are different matrix elements of the same parton operators.

This chapter is organized as follows: in Sect. 4.2 we parameterize the momenta involved in reaction (4.2) according to a longitudinal plane spanned by the photon momenta q and q'. In the next section, we provide a parameterization of the Compton tensor by means of the so-called *helicity-dependent amplitudes*, which refer to the change in the helicity of the photons. This expression of the Compton tensor is particularly useful as the helicity amplitudes can be straightforwardly related to the Compton form factors (CFFs) of the hadron. In a follow-up section, we obtain the projectors onto the different helicity amplitudes and apply them to the OPE (3.163), delivering an expression for the Compton tensor including the kinematic higher-twist components as convolutions of hard-coefficient functions with GPDs. Finally, we give full results for DDVCS and, by taking the small incoming and outgoing virtuality limits, we obtain the TCS and DVCS amplitudes, respectively. Numerical estimates of the amplitudes are also provided for the pion target.

4.2 Kinematics and longitudinal plane

In Ch. 2 we followed the parameterization of the Compton tensor and the momenta given in Ref. [42]. In that case, the longitudinal plane was spanned by the vectors

$$\bar{p} = \frac{p + p'}{2} \quad \text{and} \quad \bar{q} = \frac{q + q'}{2} \,. \tag{4.3}$$

This choice delivered photon momenta q and q' with transverse components. Consequently, the Compton tensor at LT violated the electromagnetic-gauge invariance, vid. Eq. (2.103), by terms of the order $O(\Delta_\perp/\sqrt{2\bar{p}\bar{q}}) = O(\Delta_\perp/\mathbb{Q})$ with $\mathbb{Q}^2 = Q^2 + Q'^2$. To recover gauge invariance at this approximation, we chose to evaluate the hard part of the process at $t = t_0$, which imposes $\Delta_\perp = 0$. Here, t_0 is the value of t corresponding to its minimum in absolute value. In this chapter we are interested in this kind corrections, so this evaluation is not valid anymore as it removes twist effects. However, we want to avoid an electromagnetic-gauge violating LT component. Therefore, we consider a longitudinal plane spanned by q and q', so that they do not carry transverse components:

$$n^\mu = \alpha q^\mu + \beta q'^\mu \,, \quad n'^\mu = \alpha' q^\mu + \beta' q'^\mu \,. \tag{4.4}$$

where α, β, α', β' are real parameters constrained by $n^2 = n'^2 = 0$, together with

$$nn' = -\Delta q' = \frac{t + Q^2 + Q'^2}{2} \,, \tag{4.5}$$

as well as the requirement to recover the lightlike vectors of DVCS as in [66] when approaching the limit $Q^2 \to 0$. We get the following set of equations:

$$n^2 = 0 \Rightarrow -\alpha^2 Q^2 + 2\alpha\beta(qq') + \beta^2 Q'^2 = 0 \,, \tag{4.6}$$

$$n'^2 = 0 \Rightarrow -\alpha'^2 Q^2 + 2\alpha'\beta'(qq') + \beta'^2 Q'^2 = 0 \,, \tag{4.7}$$

$$nn' = -\Delta q' \Rightarrow -\alpha\alpha' Q^2 + \beta\beta' Q'^2 + (\alpha\beta' + \alpha'\beta)(qq') + \Delta q' = 0 \,. \tag{4.8}$$

Taking into account that $\alpha \xrightarrow{Q'^2 \to 0} 0$, and choosing $\beta > 0$ as well as positive defined squared roots, we find

$$\alpha = \frac{\beta(qq' + R)}{Q^2} \,, \tag{4.9}$$

where

$$R = \sqrt{(qq')^2 + Q^2 Q'^2} \quad \text{and} \quad -qq' = \frac{t + Q^2 - Q'^2}{2} \,. \tag{4.10}$$

Likewise, considering $\alpha' \xrightarrow{Q\prime 2 \to 0} -1$, as well as selecting $\beta' > 0$:

$$\alpha' = \frac{\beta'(qq' - R)}{Q^2} \,. \tag{4.11}$$

Finally, the normalization $nn' = -\Delta q'$ gives us:

$$\beta\beta' = \frac{-(\Delta q')Q^2}{2R^2} \,, \tag{4.12}$$

which in DVCS limit renders $\beta\beta' \to Q^2/(t + Q^2)$. This last expression coincides with the value that β' should have in DVCS as in Ref. [66], so for DDVCS we might as well take

$$\beta = 1\,, \quad \beta' = \frac{-(\Delta q')Q^2}{2R^2}\,. \tag{4.13}$$

In order to avoid singularities in the TCS limit, we shift $Q^2 \times (\alpha, \beta) \to (\alpha, \beta)$ and $(1/Q^2) \times (\alpha', \beta') \to (\alpha', \beta')$. Doing so, we can write down the lightlike vectors n and n' for DDVCS as

$$n^\mu = (qq' + R)q^\mu + Q^2 q'^\mu\,, \quad n'^\mu = \frac{-(\Delta q')}{2R^2}\left(\frac{qq' - R}{Q^2}q^\mu + q'^\mu\right)\,. \tag{4.14}$$

In the limit when any of the two virtualities gets close to zero, we can apply

$$R = |qq'|\left[1 + \frac{1}{2}\frac{Q^2Q'^2}{|qq'|^2} + O\left(\frac{Q^4Q'^4}{|qq'|^4}\right)\right]\,. \tag{4.15}$$

The apparently dangerous factor is the q-component of n' in the $Q^2 \to 0$ limit, but it can be shown to be finite:

$$\frac{qq' - R}{Q^2} = \frac{qq' - \left(|qq'| + \frac{1}{2}\frac{Q^2Q'^2}{|qq'|} + O\left(\frac{Q^4Q'^4}{|qq'|^3}\right)\right)}{Q^2} \xrightarrow{Q^2 \to 0} -\frac{1}{2}\frac{Q'^2}{qq'}\,. \tag{4.16}$$

The DVCS and TCS limits of vectors (4.14) are:

$$\text{DVCS limit: } n^\mu \to Q^2 q'^\mu\,, \quad n'^\mu \to -\frac{1}{Q^2}q^\mu - \frac{1}{2(qq')}q'^\mu\,, \tag{4.17}$$

$$\text{TCS limit: } n^\mu \to 2(qq')q^\mu\,, \quad n'^\mu \to \left(\frac{1}{2(qq')} - \frac{Q'^2}{2(qq')^2}\right)\left(\frac{1}{2}\frac{Q'^2}{qq'}q^\mu - q'^\mu\right)\,. \tag{4.18}$$

Inverting relations (4.14), we find the following light-cone decomposition for the photons momenta:

$$q^\mu = \frac{1}{2R}n^\mu + \frac{RQ^2}{\Delta q'}n'^\mu\,, \quad q'^\mu = \frac{1}{2Q^2}\left(1 - \frac{qq'}{R}\right)n^\mu - R\frac{R + qq'}{\Delta q'}n'^\mu\,. \tag{4.19}$$

The apparently dangerous term is the n-component of q' as we approach the TCS case, nevertheless it turns out to be finite:

$$\frac{1}{2Q^2}\left(1 - \frac{qq'}{R}\right) = \frac{1}{2Q^2}\left(1 - \frac{qq'}{qq'}\left[1 - \frac{1}{2}\frac{Q^2Q'^2}{(qq')^2} + O\left(\frac{Q^2Q'^2}{(qq')^2}\right)\right]\right) \xrightarrow{Q^2 \to 0} \frac{1}{4}\frac{Q'^2}{(qq')^2}\,. \tag{4.20}$$

The last two vectors to decompose in their light-cone coordinates are Δ and $\bar{p}$. From q, q' above

$$\Delta^\mu = q^\mu - q'^\mu = -\frac{1}{2Q^2}\left(1 - \frac{Q^2 + qq'}{R}\right)n^\mu + \frac{R}{\Delta q'}\left(Q^2 + R + qq'\right)n'^\mu\,. \tag{4.21}$$

For $\bar{p}$, we take into account $\Delta\bar{p} = 0$, the definition of the skewness $\xi = -\Delta n/(2\bar{p}n)$ and $\bar{p}^2 = M^2 - t/4$:

$$\Delta\bar{p} = 0 \Rightarrow \bar{p}^- = \bar{p}^+ \frac{(\Delta q')(R - Q^2 - qq')}{2R^2Q^2(R + Q^2 + qq')}, \tag{4.22}$$

$$\xi = -\Delta n/(2\bar{p}n) \Rightarrow \bar{p}^+ = -\frac{R}{2\xi(\Delta q')}(R + Q^2 + qq'), \tag{4.23}$$

$$\bar{p}^2 = M^2 - t/4 \Rightarrow \bar{p}_\perp^2 = M^2 - \frac{t}{4}\left(1 - \frac{1}{\xi^2}\right). \tag{4.24}$$

We remind that for any four vector v, its light-cone coordinates are defined as: $v^\mu = v^+ n'^\mu + v^- n^\mu + v_\perp^\mu$.

Taking into account that $-\bar{p}_\perp^2 = |\bar{p}_\perp|^2 \geq 0$, one can get the value of t for which its modulus is minimal, namely t_0, for a fixed value of the skewness:

$$t_0 = -\frac{4M^2\xi^2}{1 - \xi^2} \qquad (|t| \geq |t_0|). \tag{4.25}$$

Finally, the vector $\bar{p}$ is given by

$$\begin{aligned} \bar{p}^\mu &= -\frac{R - Q^2 - qq'}{4\xi RQ^2} n^\mu - \frac{R(R + Q^2 + qq')}{2\xi(\Delta q')} n'^\mu + \bar{p}_\perp^\mu \\ &= \frac{\Delta^-}{2\xi} n^\mu - \frac{\Delta^+}{2\xi} n'^\mu + \bar{p}_\perp^\mu, \end{aligned} \tag{4.26}$$

which is well defined for both DVCS and TCS limits. Note that $\bar{p}$ is the only momentum carrying transverse components.

The skewness can be expressed through invariants. The numerator is equal to

$$\Delta n = -R(R + Q^2 + qq'), \tag{4.27}$$

while the denominator reads

$$\begin{aligned} 2\bar{p}n &= 2\bar{p}_\mu[(qq' + R)q^\mu + Q^2 q'^\mu] \\ &= (qq' + R + Q^2)2\bar{p}\bar{q} \\ &= (qq' + R + Q^2)\left(\frac{Q^2}{x_B} - \frac{Q^2 + Q'^2 - t}{2}\right). \end{aligned} \tag{4.28}$$

Here, we used $2\bar{p}\bar{q} = (\bar{p} + \bar{q})^2 - \bar{p}^2 - \bar{q}^2$, together with the conservation of momenta ($p + q = p' + q'$), $\bar{p}^2 = M^2 - t/4$, $\bar{q}^2 = -\bar{Q}^2 = -(Q^2 - Q'^2 + t/2)/2$ and the definition of the Björken variable, $x_B = Q^2/(2pq)$.

All in all, for DDVCS, ξ can be written as

$$\begin{aligned} \xi &= \frac{2R}{2Q^2/x_B - Q^2 - Q'^2 + t} \\ &= \frac{\sqrt{(Q^2 + Q'^2)^2 + t^2 + 2t(Q^2 - Q'^2)}}{2Q^2/x_B - Q^2 - Q'^2 + t}. \end{aligned} \tag{4.29}$$

Considering the DVCS limit ($Q'^2 \to 0$), this expression for the skewness takes the form given in Ref. [66].

4.3 Helicity-dependent amplitudes and the Compton tensor

Let us consider two polarization vectors for the two photons of DDVCS. We denote them as $\varepsilon^\mu(A)$ for the incoming photon, while $\varepsilon'^\mu(B)$ for the outgoing one. The labels A and B denote the polarization of the corresponding photon and take the values 0 for the longitudinal polarization (in the plane of q and q'), while ± 1 for right- and left-handed circularly polarized photon in the transverse plane (that of $\bar{p}_\perp$).

We will show that all components of the Compton tensor can be associated to helicity-dependent amplitudes. They are defined as

$$\mathcal{A}^{AB} = (-1)^{A-1} (\varepsilon'_\mu(B))^* \, T^{\mu\nu} \varepsilon_\nu(A) \,, \tag{4.30}$$

where $T^{\mu\nu}$ is the Compton tensor:

$$T^{\mu\nu} = i \int d^4z \, e^{i\bar{q}z} \langle p' | \mathbb{T}\{ j^\mu(z/2) j^\nu(-z/2) \} | p \rangle \,, \quad \bar{q} = \frac{q+q'}{2} \,. \tag{4.31}$$

These amplitudes $\mathcal{A}^{AB}$ can be interpreted as the transition amplitude from an incoming photon with polarization A to an outgoing one with polarization B.

The factor $(-1)^{A-1}$ in Eq. (4.30) comes from the following normalization of the polarization vectors:

$$\varepsilon^\mu(+)(\varepsilon_\mu(+))^* = -1 \,, \quad \varepsilon^\mu(0)\varepsilon_\mu(0) = 1 \,, \tag{4.32}$$

and likewise for the vectors ε'. Any other product vanishes. Transverse polarization vectors also satisfy

$$\varepsilon^\mu(+) = (\varepsilon^\mu(-))^* \,, \tag{4.33}$$

and the same for the vectors ε'.

For the spin-0 target, each of the helicity amplitudes allows for a parameterization of the form

$$\mathcal{A}^{AB}(\rho, \xi, t) = \bar{p}\bar{q}\mathcal{A}_1^{AB}(\rho, \xi, t) + \Delta\bar{q}\mathcal{A}_2^{AB}(\rho, \xi, t) + \cdots \,, \tag{4.34}$$

where the ellipsis accounts for coefficients $\mathcal{A}_i^{AB}$ introduced by the products $\bar{q}^2, \bar{p}^2$ and $\Delta^2 = t$. The variable ρ is a scalar made out of Minkowky products between $\bar{p}, \bar{q}$ and Δ. As a consequence, the coefficients $\mathcal{A}_i^{AB}$ described above can be combined into a single one, namely the CFF associated to the helicity change $A \to B$. Hence, for the spin-0 target, one can identify the helicity amplitude with one half of the corresponding (helicity-dependent) CFF:

$$\mathcal{A}^{AB}(\rho, \xi, t) = \frac{1}{2}\mathcal{H}^{AB}(\rho, \xi, t) \,. \tag{4.35}$$

The convenient "1/2" factor will be clear after completing the decomposition of the Compton tensor by means of helicity amplitudes.

The helicity-dependent CFFs with opposite polarizations are related by parity. This transformation acts on the states of momentum $|p\rangle$, the polarization vectors ε and the quark fields $\mathfrak{q}$ as

$$|p\rangle \text{ with } p^\mu = (p^0, \vec{p}) \to |\check{p}\rangle \text{ with } \check{p}^\mu = (p^0, -\vec{p})\,, \tag{4.36}$$

$$\varepsilon^\mu(A) \to \check{\varepsilon}^\mu(-A) = (U_P)^\mu{}_\nu \varepsilon^\nu(-A)\,, \tag{4.37}$$

$$\mathfrak{q}(z) \to \gamma^0 \mathfrak{q}(\check{z})\,, \tag{4.38}$$

where $(U_P)^\mu{}_\nu = \text{diag}(1, -1, -1, -1)$. Applying these transformations to the Compton tensor and taking into account that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \equiv (U_P)^\mu{}_\nu \gamma^\nu$ and that the time ordering is not affected by parity, it holds

$$T^{\mu\nu}(\bar{p}, \Delta, \bar{q}) \to (U_P)^\mu{}_\alpha (U_P)^\nu{}_\beta T^{\alpha\beta}(\check{\bar{p}}, \check{\Delta}, \check{\bar{q}})\,, \tag{4.39}$$

so that the helicity-dependent amplitudes transform under parity as

$$\mathcal{A}^{AB}(\rho, \xi, t) \to \mathcal{A}^{-A\,-B}(\rho, \xi, t) = \mathcal{A}^{AB}(\rho, \xi, t)\,. \tag{4.40}$$

The last equality on the RHS of the arrow is due to parity conservation. As a consequence, the total number of helicity amplitudes (CFFs) for a (pseudo-)scalar hadron is five, denoted as $\mathcal{A}^{++}$, $\mathcal{A}^{+-}$, $\mathcal{A}^{0+}$, $\mathcal{A}^{+0}$ and $\mathcal{A}^{00}$.

4.3.1 Compton tensor paramaterization by helicity amplitudes

To obtain the form of $T^{\mu\nu}$ by means of helicity-dependent amplitudes, we will make use of the spinor formalism techniques, cf. App. I, as it provides an straightforward parameterization of any tensor. See Refs. [66, 94] for further details on the formalism and its application to DVCS. We start by imposing the electromagnetic-gauge invariance:

$$q'_\mu T^{\mu\nu} = 0\,, \quad q_\nu T^{\mu\nu} = 0\,. \tag{4.41}$$

The first equation can be expressed with the dotted-undotted notation as

$$0 = \underbrace{\frac{1}{2} q'_{a\dot{a}} (\sigma_\mu)^{a\dot{a}}}_{q'_\mu} \underbrace{\frac{1}{4} T_{bc\dot{b}\dot{c}} (\sigma^\mu)^{b\dot{b}} (\sigma^\nu)^{c\dot{c}}}_{T^{\mu\nu}}\,. \tag{4.42}$$

With the notation

$$(\sigma^\mu)_{a\dot{a}} = (1, \vec{\sigma})\,, \tag{4.43}$$

where $\vec{\sigma}$ are the Pauli matrices. Making use of the following properties:

$$1)\ (\sigma^\mu)_{a\dot{a}} (\sigma_\mu)^{b\dot{b}} = 2\delta_a^b \delta_{\dot{a}}^{\dot{b}}\,, \tag{4.44}$$

$$2)\ \epsilon^{ba} q'_{a\dot{a}} \epsilon^{\dot{a}\dot{b}} = q'^{b\dot{b}}\,, \tag{4.45}$$

we may write

$$0 = 2q'^{b\dot{b}} T_{bc\dot{b}\dot{c}} (\sigma^\nu)^{c\dot{c}}\,, \tag{4.46}$$

from where we gather

$$0 = q'^{b\dot{b}} T_{bc\dot{b}\dot{c}}\,. \tag{4.47}$$

In the above formulas, summation over repeated indices is implied.

In a similar manner, the second equation in (4.41) gives

$$0 = q^{c\dot{c}} T_{bc\dot{b}\dot{c}} \,. \tag{4.48}$$

Now, we need to translate the four-vectors q, q' in (4.19) to their spin representation. In the lines of [66], let us introduce two spinors λ, μ such that

$$n_{a\dot{a}} = n^{\mu}(\sigma_{\mu})_{a\dot{a}} = \lambda_a \lambda^{\dagger}_{\dot{a}} \,, \quad n'_{a\dot{a}} = n'^{\mu}(\sigma_{\mu})_{a\dot{a}} = \mu_a \mu^{\dagger}_{\dot{a}} \,. \tag{4.49}$$

Because, n and n' are lightlike one finds the conditions:

$$\lambda\lambda = \lambda^{\dagger}\lambda^{\dagger} = \mu\mu = \mu^{\dagger}\mu^{\dagger} = 0 \,, \tag{4.50}$$

while the normalization $nn' \neq 0$ provides:

$$2nn' = (\mu\lambda)(\lambda^{\dagger}\mu^{\dagger}) = (\lambda\mu)(\mu^{\dagger}\lambda^{\dagger}) \,. \tag{4.51}$$

Hence,

$$q'^{b\dot{b}} = q'_{\mu}(\sigma^{\mu})^{b\dot{b}} = \frac{1}{2Q^2}\left(1 - \frac{qq'}{R}\right)\lambda^b \lambda^{\dagger\dot{b}} - R\frac{R + qq'}{\Delta q'}\mu^b \mu^{\dagger\dot{b}} \,, \tag{4.52}$$

and

$$q^{c\dot{c}} = \frac{1}{2R}\lambda^c \lambda^{\dagger\dot{c}} + \frac{RQ^2}{\Delta q'}\mu^c \mu^{\dagger\dot{c}} \,. \tag{4.53}$$

For the Compton tensor we can consider the general decomposition

$$T_{bc\dot{b}\dot{c}} = \lambda_b \lambda^{\dagger}_{\dot{b}} T^{(1)}_{c\dot{c}} + \mu_b \mu^{\dagger}_{\dot{b}} T^{(2)}_{c\dot{c}} + \lambda_b \mu^{\dagger}_{\dot{b}} T^{(3)}_{c\dot{c}} + \mu_b \lambda^{\dagger}_{\dot{b}} T^{(4)}_{c\dot{c}} \,, \tag{4.54}$$

for which the condition (4.42) translates to

$$T^{(2)}_{c\dot{c}} = T^{(1)}_{c\dot{c}} \frac{2R^2 Q^2}{\Delta q'} \frac{R + qq'}{R - qq'} \,. \tag{4.55}$$

Taking into account that each component $T^{(i)}_{c\dot{c}}$, $i \in \{1, 2, 3, 4\}$, can be parameterized as

$$T^{(i)}_{c\dot{c}} = \lambda_c \lambda^{\dagger}_{\dot{c}} T^{(i,1)} + \mu_c \mu^{\dagger}_{\dot{c}} T^{(i,2)} + \lambda_c \mu^{\dagger}_{\dot{c}} T^{(i,3)} + \mu_c \lambda^{\dagger}_{\dot{c}} T^{(i,4)} \,, \tag{4.56}$$

the condition (4.48) becomes

$$T^{(i,2)} = -\frac{2R^2 Q^2}{\Delta q'} T^{(i,1)} \,. \tag{4.57}$$

Introducing the relations (4.55) and (4.57) into (4.54), we get the parameterization

$$\begin{aligned} T_{bc\dot{b}\dot{c}} = &\left[\lambda_b \lambda^{\dagger}_{\dot{b}} + \frac{2R^2 Q^2}{\Delta q'} \frac{R + qq'}{R - qq'} \mu_b \mu^{\dagger}_{\dot{b}}\right] \\ &\times \left[T^{(1,2)}\left(-\frac{\Delta q'}{2R^2 Q^2}\lambda_c \lambda^{\dagger}_{\dot{c}} + \mu_c \mu^{\dagger}_{\dot{c}}\right) + T^{(1,3)}\lambda_c \mu^{\dagger}_{\dot{c}} + T^{(1,4)}\mu_c \lambda^{\dagger}_{\dot{c}}\right] \\ &+ \lambda_b \mu^{\dagger}_{\dot{b}} \times \left[T^{(1,\cdot)} \leftrightarrow T^{(3,\cdot)}\right] + \mu_b \lambda^{\dagger}_{\dot{b}} \times \left[T^{(1,\cdot)} \leftrightarrow T^{(4,\cdot)}\right] \,. \end{aligned} \tag{4.58}$$

The next step is to find the polarization vectors $\varepsilon, \varepsilon'$ and relate them to the λ, μ spinors. This can be achieved considering their orthogonality with the appropriate photon momentum, i.e.

$$q_\mu \varepsilon^\mu(A) = 0\,, \quad q'_\mu \varepsilon'^\mu(A) = 0\,, \quad A \in \{0, \pm 1\}\,, \tag{4.59}$$

as well as their normalization described in Eqs. (4.32) and (4.33).

In the context of the spin representation of the Lorentz group, one can write the Ansätz

$$\varepsilon^{a\dot{a}}(A) = \varepsilon_\mu(A)(\sigma^\mu)^{a\dot{a}} = \varepsilon^{(1)}(A)\lambda^a\lambda^{\dagger\dot{a}} + \varepsilon^{(2)}(A)\mu^a\mu^{\dagger\dot{a}} + \varepsilon^{(3)}(A)\lambda^a\mu^{\dagger\dot{a}} + \varepsilon^{(4)}(A)\mu^a\lambda^{\dagger\dot{a}}\,, \tag{4.60}$$

and likewise for ε'. One can apply it to Eq. (4.59), taking into account (4.52) and (4.53). The first equation in (4.59) provides the condition

$$\varepsilon^{(1)}(A) = -\frac{\Delta q'}{2R^2Q^2}\varepsilon^{(2)}(A)\,, \tag{4.61}$$

while the second renders

$$\varepsilon'^{(2)}(A) = \frac{2R^2Q^2}{\Delta q'}\frac{R + qq'}{R - qq'}\varepsilon'^{(1)}(A)\,. \tag{4.62}$$

Normalization (4.32) provides:

$$\begin{aligned}(-1)^A &= \frac{1}{2}\varepsilon_{a\dot{a}}(A)\varepsilon^{a\dot{a}}(-A)\\ &= \frac{1}{2}\left[-\frac{qq'}{R^2Q^2}\varepsilon^{(2)}(A)\varepsilon^{(2)}(-A) + \varepsilon^{(3)}(A)\varepsilon^{(4)}(-A) + \varepsilon^{(4)}(A)\varepsilon^{(3)}(-A)\right](\mu\lambda)(\lambda^\dagger\mu^\dagger)\,.\end{aligned} \tag{4.63}$$

Let us study the solution to the equation above for the different values of the polarization state A. For $A = +1$ and using $2nn' = (\mu\lambda)(\lambda^\dagger\mu^\dagger)$, we get

$$-\frac{1}{nn'} = -\frac{\Delta q'}{R^2Q^2}\varepsilon^{(2)}(+)\varepsilon^{(2)}(-) + \varepsilon^{(3)}(+)\varepsilon^{(4)}(-) + \varepsilon^{(4)}(+)\varepsilon^{(3)}(-)\,, \tag{4.64}$$

for which a possible solution is

$$\varepsilon^{(2)}(\pm) = 0\,, \quad \varepsilon^{(3)}(-) = -\frac{1}{\sqrt{nn'}}\,, \quad \varepsilon^{(3)}(+) = 0\,, \quad \varepsilon^{(4)}(+) = \frac{1}{\sqrt{nn'}}\,, \quad \varepsilon^{(4)}(-) = 0\,. \tag{4.65}$$

For $A = -1$, Eq. (4.64) holds and so does the solution (4.65).

Alternatively, for $A = 0$, Eq. (4.63) reduces to

$$\frac{1}{nn'} = -\frac{\Delta q'}{R^2Q^2}\varepsilon^{(2)}(0)\varepsilon^{(2)}(0) + 2\varepsilon^{(3)}(0)\varepsilon^{(4)}(0)\,, \tag{4.66}$$

for which a solution is:

$$\varepsilon^{(2)}(0) = \frac{QR}{nn'}\,, \quad \varepsilon^{(3)}(0) = \varepsilon^{(4)}(0) = 0\,. \tag{4.67}$$

For ε', the equation coming from normalization is

$$\begin{aligned}(-1)^A &= \frac{1}{2}\varepsilon'_{a\dot{a}}(A)\varepsilon'^{a\dot{a}}(-A)\\ &= \frac{1}{2}\left[\frac{4R^2Q^2}{\Delta q'}\frac{R+qq'}{R-qq'}\varepsilon'^{(1)}(A)\varepsilon'^{(1)}(-A)+\varepsilon'^{(3)}(A)\varepsilon'^{(4)}(-A)+\varepsilon'^{(4)}(A)\varepsilon'^{(3)}(-A)\right]\\ &\quad\times(\mu\lambda)(\lambda^\dagger\mu^\dagger)\,.\end{aligned} \tag{4.68}$$

We can solve it for the different values of A in a similar manner as for Eq. (4.63). For $A=\pm1$ an acceptable solution is given by the set

$$\varepsilon'^{(1)}(\pm)=0\,,\quad \varepsilon'^{(3)}(+)=0\,,\quad \varepsilon'^{(3)}(-)=-\frac{1}{\sqrt{nn'}}\,,\quad \varepsilon'^{(4)}(+)=\frac{1}{\sqrt{nn'}}\,,\quad \varepsilon'^{(4)}(-)=0\,, \tag{4.69}$$

and for $A=0$,

$$\varepsilon'^{(1)}(0)=\frac{i}{2QR}\sqrt{\frac{R-qq'}{R+qq'}}\,,\quad \varepsilon'^{(3)}(0)=\varepsilon'^{(4)}(0)=0\,. \tag{4.70}$$

We have found $\varepsilon'_{a\dot{a}}(\pm)=\varepsilon_{a\dot{a}}(\pm)$, which implies $\varepsilon'^{\mu}(\pm)=\varepsilon^{\mu}(\pm)$. This is expected as transverse polarization corresponds to the perpendicular plane and our vectors q, q' have no perpendicular components. On top of that, because $q\neq q'$ we find $\varepsilon'(0)\neq\varepsilon(0)$. Summarizing,

$$\begin{aligned}\varepsilon_{a\dot{a}}(+) &= \frac{1}{\sqrt{nn'}}\mu_a\lambda^\dagger_{\dot{a}}\,,\qquad \varepsilon_{a\dot{a}}(-)=-\frac{1}{\sqrt{nn'}}\lambda_a\mu^\dagger_{\dot{a}}\,,\\ \varepsilon_{a\dot{a}}(0) &= \frac{R}{Q(nn')}\left[\frac{nn'}{2R^2}\lambda_a\lambda^\dagger_{\dot{a}}+Q^2\mu_a\mu^\dagger_{\dot{a}}\right]\,,\\ \varepsilon'_{a\dot{a}}(0) &= \frac{i}{2RQ}\sqrt{\frac{R-qq'}{R+qq'}}\left[\lambda_a\lambda^\dagger_{\dot{a}}+\frac{2R^2Q^2}{\Delta q'}\frac{R+qq'}{R-qq'}\mu_a\mu^\dagger_{\dot{a}}\right]\,.\end{aligned} \tag{4.71}$$

Combining expressions in (4.71) with Eq. (4.58), the Compton tensor reads

$$\begin{aligned}T_{bc\dot{b}\dot{c}} &= \varepsilon'_{b\dot{b}}(0)\varepsilon_{c\dot{c}}(0)i2(\Delta q')\sqrt{\frac{R+qq'}{R-qq'}}T^{(1,2)}\\ &\quad+\varepsilon'_{b\dot{b}}(0)\varepsilon_{c\dot{c}}(-)i2R\sqrt{nn'}Q\sqrt{\frac{R+qq'}{R-qq'}}T^{(1,3)}\\ &\quad-\varepsilon'_{b\dot{b}}(0)\varepsilon_{c\dot{c}}(+)i2R\sqrt{nn'}Q\sqrt{\frac{R+qq'}{R-qq'}}T^{(1,4)}\\ &\quad+\varepsilon_{b\dot{b}}(-)\varepsilon_{c\dot{c}}(0)\frac{(\Delta q')\sqrt{nn'}}{RQ}T^{(3,2)}\\ &\quad+\varepsilon_{b\dot{b}}(-)\varepsilon_{c\dot{c}}(-)(nn')T^{(3,3)}\\ &\quad-\varepsilon_{b\dot{b}}(-)\varepsilon_{c\dot{c}}(+)(nn')T^{(3,4)}\\ &\quad-\varepsilon_{b\dot{b}}(+)\varepsilon_{c\dot{c}}(0)\frac{(\Delta q')\sqrt{nn'}}{RQ}T^{(4,2)}\\ &\quad-\varepsilon_{b\dot{b}}(+)\varepsilon_{c\dot{c}}(-)(nn')T^{(4,3)}\\ &\quad+\varepsilon_{b\dot{b}}(+)\varepsilon_{c\dot{c}}(+)(nn')T^{(4,4)}\,.\end{aligned} \tag{4.72}$$

Helicity amplitudes are defined by Eq. (4.30), so might as well write

$$T^{\mu\nu} = \sum_{A,B}(-1)^{B-1}\varepsilon'^{\mu}(B)(\varepsilon^{\nu}(A))^{*}\,\mathcal{A}^{AB} + (\text{other tensor structures})\,. \tag{4.73}$$

Quick inspection of Eq. (4.72) reveals that, for the case of DDVCS, all tensor structures can be related to a helicity amplitude, as opposed to the DVCS case [66]. Note that up to this point all expressions are general, valid for any kind of target regardless of spin, so that the particularization to a spin-0 target is relegated to the next section.

4.3.2 Spin-0 target

Particularizing for the scalar and pseudo-scalar cases, relation (4.40) applied in Eqs. (4.72) and (4.73) yields:

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{4}T_{bc\dot{b}\dot{c}}(\sigma^{\mu})^{b\dot{b}}(\sigma^{\nu})^{c\dot{c}} \\ &= -\,\mathcal{A}^{00}\varepsilon'^{\mu}(0)\varepsilon^{\nu}(0) \\ &\quad -\mathcal{A}^{+0}\left[\varepsilon'^{\mu}(0)\varepsilon^{\nu}(-) + \varepsilon'^{\mu}(0)\varepsilon^{\nu}(+)\right] \\ &\quad +\mathcal{A}^{0+}\left[\varepsilon^{\mu}(+)\varepsilon^{\nu}(0) + \varepsilon^{\mu}(-)\varepsilon^{\nu}(0)\right] \\ &\quad +\mathcal{A}^{+-}\left[\varepsilon^{\mu}(-)\varepsilon^{\nu}(-) + \varepsilon^{\mu}(+)\varepsilon^{\nu}(+)\right] \\ &\quad +\mathcal{A}^{++}\left[\varepsilon^{\mu}(+)\varepsilon^{\nu}(-) + \varepsilon^{\mu}(-)\varepsilon^{\nu}(+)\right]\,, \end{aligned} \tag{4.74}$$

Comparing the above expression with (4.72), we can relate the helicity amplitudes $\mathcal{A}^{AB}$ with the spinor components $T^{(i,j)}$. For instance:

$$\mathcal{A}^{00} = -i2(qq')\sqrt{(R+qq')/(R-qq')}\,T^{(1,2)}\,. \tag{4.75}$$

The next step is to find expressions for the polarization vectors by means of the momenta describing the longitudinal and transverse planes, i.e. $\bar{p}_{\perp}$, q and q'. We will map the vectors ε and ε' from their spin representation $\varepsilon_{a\dot{a}}$, $\varepsilon'_{a\dot{a}}$ (4.71) back to the vector one ε^{μ}, ε'^{μ}. This mapping is given by

$$\varepsilon^{\nu}(A) = \frac{1}{2}\varepsilon_{a\dot{a}}(A)(\sigma^{\mu})^{a\dot{a}}\,, \tag{4.76}$$

and likewise for ε'. Combining the above equation with the spin representation of the vectors n and n' (4.49), the longitudinal polarization vectors are immediate:

$$\varepsilon^{\nu}(0) = \frac{R}{Q(\Delta q')}\left[\frac{\Delta q'}{2R^2}n^{\nu} - Q^2 n'^{\nu}\right]\,, \tag{4.77}$$

$$\varepsilon'^{\nu}(0) = \frac{iQ}{2R}\sqrt{\frac{R-qq'}{R+qq'}}\left[\frac{1}{Q^2}n^{\nu} + \frac{2R^2}{\Delta q'}\frac{R+qq'}{R-qq'}n'^{\nu}\right]\,. \tag{4.78}$$

For the transverse polarization vectors, we need to find the combinations $\lambda_a\mu^{\dagger}_{\dot{a}}$ and $\mu_a\lambda^{\dagger}_{\dot{a}}$ in terms of momenta. To do so, we realize that they must correspond to perpendicular components of momenta as n, n' are given by $\lambda_a\lambda^{\dagger}_{\dot{a}}$ and $\mu_a\mu^{\dagger}_{\dot{a}}$, respectively. The only vector with perpendicular components is $\bar{p}$. According to App. I, we may write ($\widetilde{\bar{p}}^{\,\mu}_{\perp} = \epsilon^{\mu\nu}_{\perp}\bar{p}_\nu$):

$$\bar{p}_{\perp,b\dot{b}} = \frac{1}{(\lambda\mu)(\mu^\dagger\lambda^\dagger)}\left(p^{+-}\mu_b\lambda^\dagger_{\dot{b}} + p^{-+}\lambda_b\mu^\dagger_{\dot{b}}\right), \tag{4.79}$$

$$\widetilde{\bar{p}}_{\perp,b\dot{b}} = \frac{i}{(\lambda\mu)(\mu^\dagger\lambda^\dagger)}\left(p^{+-}\mu_b\lambda^\dagger_{\dot{b}} - p^{-+}\lambda_b\mu^\dagger_{\dot{b}}\right), \tag{4.80}$$

and, consequently,

$$\lambda_a\mu^\dagger_{\dot{a}} = \frac{\bar{p}_{\perp,a\dot{a}} + i\widetilde{\bar{p}}_{\perp,a\dot{a}}}{p^{-+}}(nn'), \quad \mu_a\lambda^\dagger_{\dot{a}} = \frac{\bar{p}_{\perp,a\dot{a}} - i\widetilde{\bar{p}}_{\perp,a\dot{a}}}{p^{+-}}(nn'). \tag{4.81}$$

Here, it was denoted $\bar{p}^{\,-+} = \bar{p}_{a\dot{a}}\mu^a\lambda^{\dagger\dot{a}}$ and $\bar{p}^{\,+-} = \bar{p}_{a\dot{a}}\lambda^a\mu^{\dagger\dot{a}}$. For the case at hand,

$$\bar{p}^{\,-+} = \bar{p}_{a\dot{a}}\sqrt{nn'}\varepsilon^{a\dot{a}}(+) = 2\sqrt{nn'}(\bar{p}\varepsilon(+)) \tag{4.82}$$

and

$$\bar{p}^{\,+-} = -\bar{p}_{a\dot{a}}\sqrt{nn'}\varepsilon^{a\dot{a}}(-) = -2\sqrt{nn'}(\bar{p}\varepsilon(-)). \tag{4.83}$$

Using $\bar{p}^{\,\mu}_{\perp} = \frac{1}{2}\bar{p}_{\perp,b\dot{b}}(\sigma^\nu)^{b\dot{b}}$ as well as the transverse polarization vectors in (4.71), we obtain

$$\bar{p}^{\,\nu}_{\perp} = -(\bar{p}\varepsilon(-))\varepsilon^\nu(+) - (\bar{p}\varepsilon(+))\varepsilon^\nu(-), \tag{4.84}$$

from where

$$\bar{p}\varepsilon(\pm) = \frac{-\bar{p}^2_\perp}{2(\bar{p}\varepsilon(\mp))} = \frac{|\bar{p}_\perp|^2}{2(\bar{p}\varepsilon(\pm))^*}. \tag{4.85}$$

Since μ, λ spinors are arbitrary and the lightlike vectors n, n' are not modified under the transformations $\mu \to e^{i\phi_1}\mu$, $\lambda \to e^{i\phi_2}\lambda$ ($\phi_1, \phi_2 \in \mathbb{R}$), then we can use this freedom to make $\bar{p}\varepsilon(\pm)$ real. Under such transformations, $\varepsilon^\nu(\pm) \to e^{\pm i(\phi_1-\phi_2)}\varepsilon^\nu(\pm)$ and $\bar{p}\varepsilon(\pm) = |\bar{p}\varepsilon(\pm)|e^{\mp i\varphi} \to |\bar{p}\varepsilon(\pm)|e^{i(\mp\varphi\pm(\phi_1-\phi_2))}$ ($\varphi \in \mathbb{R}$). We can choose $\phi_1 - \phi_2 - \varphi = 0$ so that $\bar{p}\varepsilon(\pm) \in \mathbb{R}^+$ and $\bar{p}\varepsilon(+) = \bar{p}\varepsilon(-)$. Thus,

$$\bar{p}\varepsilon(+) = \bar{p}\varepsilon(-) \Rightarrow (\bar{p}\varepsilon(\pm))^2 = \frac{|\bar{p}_\perp|^2}{2}. \tag{4.86}$$

Gathering the latest results,

$$\varepsilon^\nu(+) = \frac{1}{2}\varepsilon_{a\dot{a}}(+)(\sigma^\nu)^{a\dot{a}} = -\frac{\bar{p}^{\,\nu}_\perp - i\widetilde{\bar{p}}^{\,\nu}_\perp}{\sqrt{2}|\bar{p}_\perp|}, \tag{4.87}$$

$$\varepsilon^\nu(-) = \frac{1}{2}\varepsilon_{a\dot{a}}(-)(\sigma^\nu)^{a\dot{a}} = -\frac{\bar{p}^{\,\nu}_\perp + i\widetilde{\bar{p}}^{\,\nu}_\perp}{\sqrt{2}|\bar{p}_\perp|}. \tag{4.88}$$

Introducing the polarization vectors from Eqs. (4.77), (4.78), (4.87) and (4.88) into the Compton tensor parameterization (4.74), we obtain the following decomposition:

$$\begin{aligned} T^{\mu\nu} = {} & \mathcal{A}^{00}\frac{-i}{QQ'R^2}\left[(qq')(Q'^2q^\mu q^\nu - Q^2q'^\mu q'^\nu) + Q^2Q'^2q^\mu q'^\nu - (qq')^2q'^\mu q^\nu\right] \\ & + \mathcal{A}^{+0}\frac{i\sqrt{2}}{R|\bar{p}_\perp|}\left[Q'q^\mu - \frac{qq'}{Q'}q'^\mu\right]\bar{p}^{\,\nu}_\perp - \mathcal{A}^{0+}\frac{\sqrt{2}}{R|\bar{p}_\perp|}\bar{p}^{\,\mu}_\perp\left[\frac{qq'}{Q}q^\nu + Qq'^\nu\right] \\ & + \mathcal{A}^{+-}\frac{1}{|\bar{p}_\perp|^2}\left[\bar{p}^{\,\mu}_\perp\bar{p}^{\,\nu}_\perp - \widetilde{\bar{p}}^{\,\mu}_\perp\widetilde{\bar{p}}^{\,\nu}_\perp\right] - \mathcal{A}^{++}g^{\mu\nu}_\perp, \end{aligned} \tag{4.89}$$

Comparing the above term on $g_\perp^{\mu\nu}$

$$T^{\mu\nu} = -\mathcal{A}^{++} g_\perp^{\mu\nu} + \cdots\,, \tag{4.90}$$

with its LT approximation

$$T^{\mu\nu}|_{\mathrm{LT}} = -\frac{1}{2}\mathcal{H} g_\perp^{\mu\nu}\,, \tag{4.91}$$

adapted from Eq. (2.75) for a spin-0 target, the relation between helicity amplitudes and CFFs given in Eq. (4.35) becomes evident.

To achieve the form (4.89) of the Compton tensor, some simplifications have been made. The tensor structure multiplying $\mathcal{A}^{++}$, before writing the polarization vector in terms of momenta, is

$$\begin{aligned} T^{\mu\nu}_{(++)} &= \varepsilon^\nu(+)\varepsilon^\mu(-) + \varepsilon^\nu(-)\varepsilon^\mu(+) \\ &= \frac{1}{|\bar{p}_\perp|^2}\left(\bar{p}_\perp^\mu \bar{p}_\perp^\nu + \tilde{\bar{p}}_\perp^\mu \tilde{\bar{p}}_\perp^\nu\right) \\ &= \frac{1}{|\bar{p}_\perp|^2}\left(g_\perp^{\mu\alpha} g_\perp^{\nu\beta} + \epsilon_\perp^{\mu\alpha}\epsilon_\perp^{\nu\beta}\right)\bar{p}_\alpha \bar{p}_\beta\,. \end{aligned} \tag{4.92}$$

Let us perform a Lorentz transformation Λ to a reference frame where the lonitudinal vectors are given as in M. Diehl's review [121]. Using $\star$ to indicate the use of M. Diehl's frame we have

$$n^\mu \underset{\Lambda}{\longrightarrow} n^{\star\mu} = \frac{1}{\sqrt{2}}(1,0,0,-1) \quad \text{and} \quad n'^\mu \underset{\Lambda}{\longrightarrow} n'^{\star\mu} = \frac{1}{\sqrt{2}}(1,0,0,1) \tag{4.93}$$

as well as

$$g_\perp^{\mu\nu} \underset{\Lambda}{\longrightarrow} g_\perp^{\star\mu\nu} = \mathrm{diag}(0,-1,-1,0)\,. \tag{4.94}$$

and

$$\begin{aligned} T^{\mu\nu}_{(++)} \underset{\Lambda}{\longrightarrow} T^{\star\lambda\sigma}_{(++)} &= \Lambda^\lambda{}_\mu \Lambda^\sigma{}_\nu T^{\mu\nu}_{(++)} \\ &= \Lambda^\lambda{}_\mu \Lambda^\sigma{}_\nu \frac{1}{|\bar{p}_\perp|^2}\left(g_\perp^{\mu\alpha} g_\perp^{\nu\beta} + \epsilon_\perp^{\mu\alpha}\epsilon_\perp^{\nu\beta}\right)\bar{p}_\alpha \bar{p}_\beta \\ &= \Lambda^\lambda{}_\mu \Lambda^\sigma{}_\nu \frac{1}{|\bar{p}_\perp|^2}\left(g_\perp^{\mu\alpha} g_\perp^{\nu\beta} + \epsilon_\perp^{\mu\alpha}\epsilon_\perp^{\nu\beta}\right)(\Lambda^T)_\alpha{}^\delta (\Lambda^T)_\beta{}^\Delta \bar{p}^\star_\delta \bar{p}^\star_\Delta \\ &= \frac{1}{|\bar{p}_\perp|^2}\left(g_\perp^{\star\lambda\delta} g_\perp^{\star\sigma\Delta} + \epsilon_\perp^{\star\lambda\delta}\epsilon_\perp^{\star\sigma\Delta}\right)\bar{p}^\star_\delta \bar{p}^\star_\Delta\,. \end{aligned} \tag{4.95}$$

Here, we used Λ^T for the inverse transformation (given as transpose matrix), $g_\perp^{\star\lambda\delta} = g^{\lambda\delta} - \frac{1}{nn'}(n^{\star\lambda}n'^{\star\delta} + n'^{\star\lambda}n^{\star\delta})$ and $n^{\star\lambda} = \Lambda^\lambda{}_\alpha n^\alpha$ (similarly for $n'^\star$ and $\epsilon_\perp^{\star\lambda\delta}$).

Studying component by component, one can show that

$$T^{\star\lambda\sigma}_{(++)} = -g_\perp^{\star\lambda\sigma}\,, \tag{4.96}$$

and using the inverse Lorentz transformation to return to the longitudinal plane described in Sect. 4.2, we get ($n^\star \underset{\Lambda^T}{\longrightarrow} n$, $n'^\star \underset{\Lambda^T}{\longrightarrow} n'$):

$$T^{\mu\nu}_{(++)} = -g_\perp^{\mu\nu}\,. \tag{4.97}$$

Projectors onto helicity amplitudes

Close inspection of expression (4.89) allows you to define a set of helicity projectors $\Pi^{(AB)}_{\mu\nu}$ such that

$$\Pi^{(AB)}_{\mu\nu} T^{\mu\nu} = \mathcal{A}^{AB} . \tag{4.98}$$

Considering $\bar{p}_{\perp,\alpha} q^{\alpha} = \bar{p}_{\perp,\alpha} q'^{\alpha} = 0$, one can define

$$\Pi^{(0+)}_{\mu\nu} = \bar{p}_{\perp,\mu} q'_{\nu} \frac{R}{\sqrt{2}|\bar{p}_{\perp}|} \left[QQ'^2 + \frac{(qq')^2}{Q} \right]^{-1} = \bar{p}_{\perp,\mu} q'_{\nu} \frac{Q}{\sqrt{2}|\bar{p}_{\perp}|R} , \tag{4.99}$$

$$\Pi^{(+0)}_{\mu\nu} = -q_{\mu} \bar{p}_{\perp,\nu} \frac{iR}{\sqrt{2}|\bar{p}_{\perp}|} \left[Q'Q^2 + \frac{(qq')^2}{Q'} \right]^{-1} = -q_{\mu} \bar{p}_{\perp,\nu} \frac{iQ'}{\sqrt{2}|\bar{p}_{\perp}|R} . \tag{4.100}$$

Using $\widetilde{\bar{p}}^{\,2}_{\perp} = \bar{p}^{\,2}_{\perp} = -|\bar{p}_{\perp}|^2$:

$$\Pi^{(++)}_{\mu\nu} = \frac{1}{2|\bar{p}_{\perp}|^2} \left(\bar{p}_{\perp,\mu} \bar{p}_{\perp,\nu} + \widetilde{\bar{p}}_{\perp,\mu} \widetilde{\bar{p}}_{\perp,\nu} \right) = -\frac{g_{\perp,\mu\nu}}{2} , \tag{4.101}$$

and with $\bar{p}_{\perp} \cdot \widetilde{\bar{p}}_{\perp} = \bar{p}_{\perp,\mu} \epsilon^{\mu\nu}_{\perp} \bar{p}_{\perp,\nu} = 0$:

$$\Pi^{(+-)}_{\mu\nu} = \frac{1}{2|\bar{p}_{\perp}|^2} \left(\bar{p}_{\perp,\mu} \bar{p}_{\perp,\nu} - \widetilde{\bar{p}}_{\perp,\mu} \widetilde{\bar{p}}_{\perp,\nu} \right) . \tag{4.102}$$

Finally, for amplitude $\mathcal{A}^{00}$ we can only use a projector made out of longitudinal vectors, i.e. q, q'. We can make it antisymmetric to obtain the simplest of the projectors. This way,

$$\Pi^{(00)}_{\mu\nu} = -\frac{i2QQ'}{R^2} q_{[\mu} q'_{\nu]} , \tag{4.103}$$

where $q_{[\mu} q'_{\nu]} = (q_{\mu} q'_{\nu} - q_{\nu} q'_{\mu})/2$.

In what follows, we will use this projectors to compute the different helicity amplitudes from the Compton tensor (3.163).

4.4 Transverse-helicity conserving amplitude, $\mathcal{A}^{++}$

In the current section we present the calculation of the transverse helicity-conserving amplitude $\mathcal{A}^{++}$ up to kinematic twist-4 accuracy. To do so, we will use the Compton tensor decomposition in terms of the light-ray operators $\mathscr{O}$ (3.163) and apply the projector $\Pi^{(++)}_{\mu\nu}$ from Eq. (4.101).

Amplitude $\mathcal{A}^{++}$ will be the only one with a contribution at kinematic leading-twist (twist-2) as its corresponding Lorentz structure is the transverse metric, $g^{\mu\nu}_{\perp}$, vid. Eq. (4.89).

Applying projector $\Pi^{(++)}_{\mu\nu}$ to $T^{\mu\nu}$ (3.163), all terms proportional to q, q' or Δ are washed out. Antisymmetric terms drop as well.

In what follows, we will show the calculation of $\mathcal{A}^{++}$ integral by integral, starting with the first line of Compton tensor (3.163). In this line, we find the most singular term ($\sim 1/z^4$) so that it must contain the usual leading twist contribution to the amplitude of DDVCS. This is,

$$\mathcal{A}^{++}\big|_{\rm LT} \subset -\frac{g_{\perp,\mu\nu}}{2i\pi^2} i \int d^4z\, e^{iq'z} \frac{1}{(-z^2+i0)^2} \Bigg[g^{\mu\nu} \mathscr{O}(1,0) - z^\nu \partial^\mu \int_0^1 du\, \mathscr{O}(\bar{u},0) - z^\mu \partial^\nu \int_0^1 dv\, \mathscr{O}(1,v) \Bigg] . \tag{4.104}$$

Introducing the expression (3.188) for the matrix elements of the light-ray operators, we are left with the following Fourier transforms:

$$i \int d^4z\, e^{iq'z} \frac{\left[e^{-i\ell z}\right]_{\rm LT}}{(-z^2+i0)^2} , \quad i \int d^4z\, e^{iq'z} \frac{z^\alpha \partial^\beta \left[e^{-i\ell z}\right]_{\rm LT}}{(-z^2+i0)^2} , \tag{4.105}$$

where ℓ is a shorthand for the general

$$\ell_{\lambda_1,\lambda_2} = -\lambda_1 \Delta - \lambda_{12} \left[\beta \bar{p} - \frac{1}{2}(\alpha+1)\Delta \right] . \tag{4.106}$$

Here, $[\,]_{\rm LT}$ stands for the geometric LT projection given by Eq. (D.19). With said projector, the first integral yields

$$\begin{aligned}
& i \int d^4z\, e^{iq'z} \frac{1}{(-z^2+i0)^2} \left[e^{-i\ell z}\right]_{\rm LT} = \\
&\quad = i \int d^4z\, \frac{1}{(-z^2+i0)^2} \left[e^{i(q'-\ell)z} + \int_0^1 dw\, \frac{z^2\ell^2}{4} w\, e^{i(q'-w\ell)z} \right] \\
&\quad = I_2^{(4+2\varepsilon)}\Big|_{r=q'-\ell} - \frac{\ell^2}{4} \int_0^1 dw\, w\, I_1^{(4)}\Big|_{r=q'-w\ell} \\
&\quad = -\pi^2 \ln\left(\frac{\ell^2 - 2q'\ell + Q'^2 + i0}{-\mu^2} \right) + \pi^2 \ell^2 I_{1,1} \\
&\quad = -\pi^2 \ln\left(\frac{a+b+c}{-\mu^2} \right) + \frac{\pi^2}{2} \ln\left(\frac{a+b+c}{c} \right) \\
&\qquad + \frac{\pi^2 b}{2\mathscr{R}} \left[\ln\left(1 + \frac{b-\mathscr{R}}{2c} \right) - \ln\left(1 + \frac{b+\mathscr{R}}{2c} \right) \right] .
\end{aligned} \tag{4.107}$$

We introduced the notation $I_1^{(4)}$ and $I_2^{(4+2\epsilon)}$ for two Fourier transforms detailed in App. J. The upper index indicates the dimension in which the Fourier transform is performed. In the case of $I_2^{(4+2\epsilon)}$, the procedure of dimensional regularization has been used with a positive $\epsilon \to 0$. Also, we named $I_{1,1}$ to the following integral:

$$I_{1,1} = \int_0^1 dw\, \frac{w}{aw^2 + bw + c} , \tag{4.108}$$

where

$$a = \ell^2 , \quad b = -2q'\ell , \quad c = Q'^2 + i0 . \tag{4.109}$$

The result of the integral $I_{1,1}$ is given by means of the additional variable $\mathscr{R}$, defined as

$$\mathscr{R} = \sqrt{b^2 - 4ac} . \tag{4.110}$$

In general, we will denote the integrals $I_{n,m}$ as

$$\int_0^1 dw \, \frac{w^n}{(aw^2+bw+c)^m} \, . \tag{4.111}$$

We also drop any constant terms as they vanish upon integration with $\Phi^{(+)}$. We will consider this as a rule whenever a constant is presented, unless stated otherwise. In addition, in the Fourier transform above, we cut the LT expansion of the exponential to z^2 power since

$$\frac{z^2}{-z^2+i0} = -\frac{z^4}{z^4+0^2} - i\pi z^2\delta(z^2) \underset{\int d^4z}{\longmapsto} -\frac{z^4}{z^4+0^2} = -1 \, . \tag{4.112}$$

The last equality is a consequence of considering the limit as $z^4 \to 0$ by L'Hôpital. This observation suggests that in $\left[e^{-i\ell z}\right]_{\mathrm{LT}}$, only the terms that do not fully compensate the light-cone divergence in z^2 are to be accounted for. Positive powers of z^2 produce Dirac deltas and derivatives of those that vanish upon integration with respect to w.

The second integral in (4.107) can be done by parts:

$$i\int d^4z \, e^{iq'z} \frac{z^\alpha \partial^\beta \left[e^{-i\ell z}\right]_{\mathrm{LT}}}{(-z^2+i0)^2} = -\, i\int d^4z \, e^{iq'z} \frac{1}{(-z^2+i0)^2} \Big[iq'^\beta z^\alpha + g^{\alpha\beta} + 4\frac{z^\alpha z^\beta}{-z^2+i0} \Big] \left[e^{-i\ell z}\right]_{\mathrm{LT}} . \tag{4.113}$$

For $\mathcal{A}^{++}$, the term in q' does not contribute so we will ignore it hereafter. From the expression above, we need:

$$\begin{aligned}
i\int d^4z \, e^{iq'z} & \frac{z^\nu z^\mu}{(-z^2+i0)^3} \left[e^{-i\ell z}\right]_{\mathrm{LT}} = \\
&= i\int d^4z \, e^{iq'z} \frac{z^\nu z^\mu}{(-z^2+i0)^3} \left[e^{-i\ell z} + \int_0^1 dw \, \frac{z^2\ell^2}{4} \, w \, e^{-iw\ell z} + \int_0^1 dw \, \frac{z^4\ell^4}{4^2 2} \, \bar{w} \, w^2 e^{-iw\ell z} \right] \\
&= \widetilde{I}_3^{(4+2\varepsilon),\,\mu\nu}\Big|_{r=q'-\ell} - \frac{\ell^2}{4}\int_0^1 dw \, w \, \widetilde{I}_2^{(4+2\varepsilon),\,\mu\nu}\Big|_{r=q'-w\ell} + \frac{\ell^4}{4^2 2}\int_0^1 dw \, \bar{w} \, w^2 \, \widetilde{I}_1^{(4+2\varepsilon),\,\mu\nu}\Big|_{r=q'-w\ell} \\
&= \frac{\pi^2}{4} g^{\mu\nu} \ln\left(\frac{a+b+c}{-\mu^2}\right) + \frac{\pi^2}{2} \frac{(q'-\ell)^\nu (q'-\ell)^\mu}{a+b+c} \\
&\quad - \frac{\ell^2\pi^2}{2}\left[I_{1,1} g^{\mu\nu} - 2I_{3,2}\ell^\nu\ell^\mu - 2I_{1,2}q'^\nu q'^\mu + 4I_{2,2}q'^{(\mu}\ell^{\nu)} \right] \\
&\quad - \frac{\ell^4\pi^2}{4}\Big[(I_{2,2}-I_{3,2})g^{\mu\nu} - 4(I_{4,3}-I_{5,3})\ell^\nu\ell^\mu - 4(I_{2,3}-I_{3,3})q'^\nu q'^\mu \\
&\qquad - 8(I_{3,3}-I_{4,3})q'^{(\mu}\ell^{\nu)} \Big] \, .
\end{aligned} \tag{4.114}$$

To take care of the twist expansion we need to identify the scale of DDVCS, which thus far was taken *ad-hoc* as $\mathbb{Q}^2 = Q^2 + Q'^2$. The choice of this scale was unclear due to the $\bar{q}$ dependence of the Compton tensor and led to some discussion, cf. [76]. The structure of the integrals $I_{n,m}$ that appear in the Fourier transforms (4.107) and

(4.114), together with the expressions

$$\begin{aligned}
a &= \ell^2 = \lambda_1^2 t + \lambda_{12}^2 \left[\beta^2 \bar{p}^2 + \frac{1}{4}(\alpha+1)^2 t\right] - \lambda_1\lambda_{12}(\alpha+1)t\,,\\
b &= -2q'\ell = -2(q'\Delta)\left[\lambda_{12}\frac{\beta}{2\Omega} + \frac{\lambda_{12}}{2}(\alpha+1) - \lambda_1\right],\\
\Omega &= -\frac{\Delta q'}{2\bar{p}q'}\,,\\
c &= Q'^2 + i0\,,
\end{aligned} \tag{4.115}$$

suggests the scale

$$\mathbb{Q}^2 = -2q'\Delta = Q^2 + Q'^2 + t\,, \tag{4.116}$$

which coincides with the variable chosen up to this point, apart from a factor t. Since the twist expansion happens naturally in the variable $-2q'\Delta$ (4.116) and the difference between keeping and removing t is a higher twist, we decide to use the scale $\mathbb{Q}^2 = -2q'\Delta$. Consequently, the kinematic-twist expansion shall be done in powers of

$$\frac{a}{b} = O\left(\frac{|t|}{\mathbb{Q}^2}, \frac{M^2}{\mathbb{Q}^2}\right) = O(\text{tw-4}) \quad \text{and} \quad \frac{c}{b} = O\left(\frac{Q'^2}{\mathbb{Q}^2}\right) = O(1)\,. \tag{4.117}$$

4.4.1 Power expansion of the Fourier transform in Eq. (4.107)

First, consider the factor $b/\mathscr{R}$ up to twist-4, i.e. keeping up to powers $(a/b)^1$ and $(c/b)^1$:

$$\frac{b}{\mathscr{R}} = \frac{b}{b\sqrt{1-\frac{4ac}{b^2}}} = \sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{-4ac}{b^2}\right)^n = 1 + \frac{2ac}{b^2} + O(\text{tw-6})\,. \tag{4.118}$$

Secondly, consider the following combination of logarithms:

$$\ln\left(1+\frac{b-\mathscr{R}}{2c}\right) - \ln\left(1+\frac{b+\mathscr{R}}{2c}\right) + \ln\left(\frac{a+b+c}{c}\right) = 2\ln\left(1+\frac{b-\mathscr{R}}{2c}\right). \tag{4.119}$$

This last expression can be shown to be a twist-4 contribution. The argument of the logarithm can be expanded in Taylor series as

$$1+\frac{b-\mathscr{R}}{2c} = 1 - \frac{b}{2c}\sum_{n=1}^{\infty}\binom{1/2}{n}\left(\frac{-4ac}{b^2}\right), \tag{4.120}$$

which renders the following expression for the logarithm in Eq. (4.119):

$$\begin{aligned}
\ln\left(1+\frac{b-\mathscr{R}}{2c}\right) &= \sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\left(\frac{-b}{2c}\sum_{n=1}^{\infty}\binom{1/2}{n}\left(\frac{-4ac}{b^2}\right)^n\right)^{m+1}\\
&= \underbrace{\frac{a}{b}}_{O(\text{tw-4})} + O(\text{tw-6})\,.
\end{aligned} \tag{4.121}$$

It is indeed a twist-4 contribution.

There is still another logarithm in Eq. (4.107):

$$\ln\left(\frac{a+b+c}{-\mu^2}\right) = \ln\left(\frac{b+c}{-\mu^2}\left(1+\frac{a}{b+c}\right)\right) = \underbrace{\ln\left(\frac{b+c}{-\mu^2}\right)}_{O(1)} + \underbrace{\frac{a}{b+c}}_{O(\text{tw-4})} + O(\text{tw-6})\,. \tag{4.122}$$

Gathering results thus far, the twist expansion of the Fourier transform of $1/z^4$ as given in Eq. (4.107) takes the form:[1]

$$\begin{aligned} i\int d^4z\, \frac{e^{iq'z}\left[e^{-i\ell z}\right]_{\text{LT}}}{(-z^2+i0)^2} =& -\pi^2\ln\left(\frac{b+c}{-\mu^2}\right) - \pi^2\frac{a}{b+c} + \pi^2\frac{a}{b} \\ &+ \pi^2\frac{ac}{b^2}\left[\ln\left(1+\frac{b-\mathscr{R}}{2c}\right) - \ln\left(1+\frac{b+\mathscr{R}}{2c}\right)\right] + O(\text{tw-6})\,. \end{aligned} \tag{4.123}$$

The remnant logarithms in the above Fourier transform can be further expanded:

$$\ln\left(1+\frac{b-\mathscr{R}}{2c}\right) - \ln\left(1+\frac{b+\mathscr{R}}{2c}\right) = \underbrace{\ln\left(\frac{c}{b+c}\right)}_{O(1)} + \underbrace{2\ln\left(1+\frac{b-\mathscr{R}}{2c}\right) - \ln\left(1+\frac{a}{b+c}\right)}_{O(\text{tw-4})}\,. \tag{4.124}$$

This expression appears multiplied by a factor $ac/b^2 = O(\text{tw-4})$, hence to a global twist-4 accuracy we only need to keep the first logarithm. Therefore, we conclude:

$$i\int d^4z\, \frac{e^{iq'z}\left[e^{-i\ell z}\right]_{\text{LT}}}{(-z^2+i0)^2} = -\pi^2\ln\left(\frac{b+c}{-\mu^2}\right) - \pi^2\frac{a}{b+c} + \pi^2\frac{a}{b} + \pi^2\frac{ac}{b^2}\ln\left(\frac{c}{b+c}\right) + O(\text{tw-6})\,. \tag{4.125}$$

4.4.2 Power expansion of Fourier transform in Eq. (4.114)

$$\begin{aligned} i\int d^4z\, e^{iq'z}\frac{z^\nu z^\mu}{(-z^2+i0)^3}\left[e^{-i\ell z}\right]_{\text{LT}} =& \frac{\pi^2}{4}g^{\mu\nu}\left[\ln\left(\frac{b+c}{-\mu^2}\right) + \frac{a}{b+c}\right] + \frac{\pi^2}{2}\frac{\ell^\nu\ell^\mu}{b+c} \\ &- \frac{\ell^2\pi^2}{2}\left[I_{1,1}g^{\mu\nu} - 2I_{3,2}\ell^\nu\ell^\mu\right]\Big|_{\text{LT+tw-4}} \\ &- \frac{\ell^4\pi^2}{4}\left[(I_{2,2}-I_{3,2})g^{\mu\nu} - 4(I_{4,3}-I_{5,3})\ell^\nu\ell^\mu\right]\Big|_{\text{LT+tw-4}} \\ &+ (\text{terms} \sim q') + O(\text{tw-6})\,. \end{aligned} \tag{4.126}$$

Let us break down the different terms with $I_{n,m}$ into their twist components. Here, we consider $\ell^\nu\ell^\mu/b = O(\text{tw-4})$ because in the context of $\mathcal{A}^{++}$, $\ell^\nu\ell^\mu$ will be contracted with $g_{\perp,\mu\nu}$ eventually. Therefore $g_{\perp,\mu\nu}\ell^\nu\ell^\mu/b = \ell_\perp^2/b = O(\bar{p}_\perp^2/\mathbb{Q}^2) =$

[1] We omit constant terms as they will vanish upon integration with the DD $\Phi^{(+)}$ due to Eq. (3.186).

$O(|t|/\mathbb{Q}^2, M^2/\mathbb{Q}^2)$. Likewise, $\ell^\nu \ell^\mu / a = O(1)$. Then, it holds:

$$\ell^2 I_{1,1} = \frac{a}{b} + \frac{ac}{b^2} \ln\left(\frac{c}{b+c}\right) + O(\text{tw-6}), \tag{4.127}$$

$$\ell^\nu \ell^\mu \frac{b^2 - 2ac + bc}{(a+b+c)\mathscr{R}^2} = \frac{\ell^\nu \ell^\mu}{b} + O(\text{tw-6}), \tag{4.128}$$

while

$$\ell^2 \ell^\nu \ell^\mu I_{3,2}\,, \quad \ell^4 I_{2,2}\,, \quad \ell^4 \frac{b+2c}{\mathscr{R}^2(a+b+c)}\,, \quad \ell^4 \ell^\nu \ell^\mu I_{4,3} \quad \text{and} \quad \ell^4 \ell^\nu \ell^\mu I_{5,3} \tag{4.129}$$

start at kinematic twist-6.

Finally, introducing this power expansion in Eq. (4.126) we find

$$\begin{aligned} i \int d^4 z\, e^{iq'z} \frac{z^\nu z^\mu}{(-z^2 + i0)^3} \left[e^{-i\ell z}\right]_{\text{LT}} &= \frac{\pi^2}{2} g^{\mu\nu} \left\{ \frac{1}{2} \ln\left(\frac{b+c}{-\mu^2}\right) + \frac{a}{2(b+c)} - \frac{a}{b} - \frac{ac}{b^2} \ln\left(\frac{c}{b+c}\right) \right\} \\ &+ \frac{\pi^2}{2} \frac{\ell^\nu \ell^\mu}{b+c} + (\text{terms} \sim q') + O(\text{tw-6}). \end{aligned} \tag{4.130}$$

Taking into account the result (4.125), we encounter:

$$\begin{aligned} i \int d^4 z\, e^{iq'z} \frac{z^\nu \partial^\mu}{(-z^2 + i0)^2} \left[e^{-i\ell z}\right]_{\text{LT}} &= g^{\mu\nu} \pi^2 \left[\frac{a}{b} + \frac{ac}{b^2} \ln\left(\frac{c}{b+c}\right) \right] - 2\pi^2 \frac{\ell^\nu \ell^\mu}{b+c} \\ &+ (\text{terms} \sim q') + O(\text{tw-6}), \end{aligned} \tag{4.131}$$

which is a pure twist-4 contribution. We conclude that the LT will be found in the term $\sim g^{\mu\nu}$ of Eq. (4.104).

4.4.3 Leading-twist contribution and definition of the generalized Björken variable

From the Fourier transforms computed previously, the leading-twist contribution to the amplitude is provided by the first logarithm in Eq. (4.125):

$$\begin{aligned} \mathcal{A}^{++}|_{\text{LT}} &\subset -\frac{g_{\perp,\mu\nu}}{2i\pi^2} 2i \iint_{\mathbb{D}} d\beta d\alpha\, \Phi^{(+)}\, g^{\mu\nu} \left[-\pi^2 \ln\left(\frac{b+c}{-\mu^2}\right) \right] \bigg|_{\ell=\ell_{1,0}} \\ &\subset \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)} \ln\left(\frac{b+c}{-\mu^2}\right) \bigg|_{\ell=\ell_{1,0}}, \end{aligned} \tag{4.132}$$

where $\Phi^{(+)} = \Phi^{(+)}(\beta, \alpha, t)$.

To map the DD $\Phi^{(+)}$ to the corresponding GPD $H^{(+)}$ we need to introduce the identity $1 = \int_{-1}^{1} dx\, \delta(x - \beta - \alpha\xi)$ in $\mathcal{A}^{++}$. This identity leads to

$$\begin{aligned} b &= -2q'\ell = \mathbb{Q}^2 \left[\lambda_{12} \left(\frac{x+\xi}{2\xi} + \beta \frac{\rho - \xi}{2\xi^2} \frac{t}{\mathbb{Q}^2} \right) - \lambda_1 + O(\text{tw-6}) \right], \\ c &= Q'^2 + i0, \end{aligned} \tag{4.133}$$

for operator $\mathscr{O}(\lambda_1, \lambda_2)$ and $\lambda_{12} = \lambda_1 - \lambda_2$. With (4.133):

$$\begin{aligned}\mathcal{A}^{++}\big|_{\text{LT}} &\subset \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi)\, \ln\left(\frac{b+c}{-\mu^2}\right)\bigg|_{\ell=\ell_{1,0}} \\ &\subset \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi) \\ &\quad \times \ln\left(\frac{\mathbb{Q}^2}{-2\mu^2}\left[\frac{x-\xi}{\xi}+\frac{2Q'^2}{\mathbb{Q}^2}+i0\right]\left[1+\frac{\beta\left(\frac{1}{\Omega}-\frac{1}{\xi}\right)}{\frac{x-\xi}{\xi}+\frac{2Q'^2}{\mathbb{Q}^2}+i0}\right]\right). \end{aligned} \tag{4.134}$$

Let us show that the β-term is actually a twist-4. From its definition in (4.115), one can show that Ω is directly proportional to the skewness ξ:

$$\Omega = \xi\frac{\mathbb{Q}^2}{2R}. \tag{4.135}$$

From Eq. (4.29), $2R$ reads

$$\begin{aligned} 2R &= \sqrt{(Q^2+Q'^2)^2+t^2+2t(Q^2-Q'^2)} \\ &= (Q^2+Q'^2)\sum_{n=0}^{\infty}\binom{1/2}{n}\left(\frac{t^2}{(Q^2+Q'^2)^2}+\frac{2t(Q^2-Q'^2)}{(Q^2+Q'^2)^2}\right)^n, \end{aligned} \tag{4.136}$$

which can be used to compute

$$\frac{1}{\Omega}-\frac{1}{\xi} = \frac{1}{\xi}\underbrace{\left(\frac{-2tQ'^2}{\mathbb{Q}^4}\right)}_{O(\text{tw-4})} + O(\text{tw-6}), \tag{4.137}$$

where we promoted $Q^2+Q'^2 \to \mathbb{Q}^2$ in the denominator, as the difference is a twist-6. Since β, x, ξ and $Q'^2/\mathbb{Q}^2$ are order one parameters, we conclude that

$$\frac{\beta\left(\frac{1}{\Omega}-\frac{1}{\xi}\right)}{\frac{x-\xi}{\xi}+\frac{2Q'^2}{\mathbb{Q}^2}+i0} = O\left(\frac{|t|Q'^2}{\mathbb{Q}^4}\right) = O(\text{tw-4}). \tag{4.138}$$

Hence, going back to the amplitude:

$$\begin{aligned}\mathcal{A}^{++}\big|_{\text{LT}} &\subset \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi)\left\{\ln\left(\frac{x-\xi}{\xi}+\frac{2Q'^2}{\mathbb{Q}^2}+i0\right)\right. \\ &\quad \left. -\beta\frac{2Q'^2 t}{\mathbb{Q}^4\left(x-\xi+\xi\frac{2Q'^2}{\mathbb{Q}^2}+i0\right)}\right\} \\ &\quad + \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi)\ln\left(\frac{\mathbb{Q}^2}{-2\mu^2}\right). \end{aligned} \tag{4.139}$$

Employing the relation (3.192) we find the last term to vanish:

$$\begin{aligned}
&\int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi)\ln\left(\frac{Q^2}{-2\mu^2}\right) = \\
&= \frac{1}{2}\int_{-1}^{1} dx\, \ln\left(\frac{Q^2}{-2\mu^2}\right)\partial_x H^{(+)} \\
&= \frac{1}{2}\ln\left(\frac{Q^2}{-2\mu^2}\right)\left[H^{(+)}(1,\xi,t) - H^{(+)}(-1,\xi,t)\right] \\
&= 0\,,\ \text{since } H^{(+)}(\pm 1,\xi,t) = 0\,. \qquad (4.140)
\end{aligned}$$

The β-term in Eq. (4.139) is clearly a twist-4 component. To the LT we can omit this factor. The structure of this solution motivates the definition of a new variable named *generalized Björken variable* and denoted as ρ:

$$\frac{x-\xi}{\xi} + \frac{2Q'^2}{Q^2} = \frac{x-\rho}{\xi} \Rightarrow \rho = \xi\frac{qq'}{\Delta q'}\,, \qquad (4.141)$$

which justifies the definition given in Chs. 1 and 2. By means of the kinematics, ρ takes the following form:

$$\begin{aligned}
\rho &= \xi\frac{Q^2 - Q'^2 + t}{Q^2 + Q'^2 + t} \\
&= \xi_{[42]} + \frac{\xi t}{2Q^2} + O(\text{tw-6})\,. \qquad (4.142)
\end{aligned}$$

The variable $\xi_{[42]}$ is the generalized Björken variable as introduced by Belitsky and Müller in Ref. [42]:

$$\xi_{[42]} = \xi\frac{2\bar{Q}^2}{Q^2 + Q'^2}\,, \quad \bar{Q}^2 = -\bar{q}^2 = \frac{1}{2}\left(Q^2 - Q'^2 + \frac{t}{2}\right)\,. \qquad (4.143)$$

To avoid confusion with the skewness,[2] we denote it as ρ following notation in [121]. To LT accuracy all definitions agree. The definition of the generalized Björken variable given in Eq. (4.141) comes from formulating the twist expansion of the scattering amplitude and, therefore, it is valid beyond the LT approximation.

Introducing ρ in the amplitude (4.139), to kinematic twist-2 accuracy we have:

$$\begin{aligned}
\mathcal{A}^{++}\big|_{\text{LT}} &= \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi)\ln\left(\frac{x-\rho}{\xi} + i0\right) \\
&= \int_{-1}^{1} dx\, \frac{1}{2}\left(\partial_x H^{(+)}\right)\ln\left(\frac{x-\rho}{\xi} + i0\right) \\
&= -\int_{-1}^{1} dx\, H^{(+)}\frac{1}{2}\frac{1/\xi}{\frac{x-\rho}{\xi} + i0} \\
&= \frac{1}{2}\int_{-1}^{1} dx\, \frac{1}{\rho - x - i0}H^{(+)}(x,\xi,t)\,, \qquad (4.144)
\end{aligned}$$

which agrees with the usual LT approximation, see for example [42, 43, 69]. In the integration by parts, the boundary terms vanish as a consequence of $H^{(+)}(\pm 1,\xi,t) = 0$.

[2] In Ref. [42], another variable denoted as $-\eta$ is used as the skewness. It coincides with the one used here and defined in Eq. (1.65) only at LT.

4.4.4 Twist-4 contribution

The next step is to gather all contributions to twist-4 for $\mathcal{A}^{++}$. We continue with the first line of the Compton tensor (3.163), this is

$$\mathcal{A}^{++} \supset -\frac{g_{\perp,\mu\nu}}{2i\pi^2} i \int d^4z\, e^{iq'z} \frac{1}{(-z^2+i0)^2} \left[g^{\mu\nu} \mathscr{O}(1,0) - z^\nu \partial^\mu \int_0^1 du\, \mathscr{O}(\bar{u},0) \right. \\ \left. - z^\mu \partial^\nu \int_0^1 dv\, \mathscr{O}(1,v) \right] . \tag{4.145}$$

The twist-4 part of this expression consists of the β-term found in Eq. (4.139), which accounts for the twist-4 component of the term proportional to the metric in (4.145). We will denote this contribution as $I_{(0)}$. The contribution from the terms $\sim z^\nu \partial^\mu \mathscr{O}(\bar{u},0)$, $z^\mu \partial^\nu \mathscr{O}(1,v)$ is delivered by the Fourier transform (4.131) and we will denote it as $I_{(\mathrm{i})} + I_{(\mathrm{ii})}$. Hence,

$$\mathcal{A}^{++}|_{\text{tw-4}} \supset I_{(0)} + I_{(\mathrm{i})} + I_{(\mathrm{ii})} , \tag{4.146}$$

where

$$I_{(0)} = -\frac{t}{Q^2}\frac{\xi-\rho}{\xi} \int_{-1}^1 dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)} \delta(x-\beta-\alpha\xi) \frac{\beta}{x-\rho+i0} , \tag{4.147}$$

$$I_{(\mathrm{i})} = \int_{-1}^1 dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)} \delta(x-\beta-\alpha\xi) \int_0^1 \frac{du}{\bar{u}} \left[\frac{a}{b} + \frac{ac}{b^2} \ln\left(\frac{c}{b+c}\right) - \frac{\ell_\perp^2}{b+c} \right]\Bigg|_{\ell=\ell_{\bar{u},0}} , \tag{4.148}$$

$$I_{(\mathrm{ii})} = \int_{-1}^1 dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)} \delta(x-\beta-\alpha\xi) \int_0^1 \frac{dv}{\bar{v}} \left[\frac{a}{b} + \frac{ac}{b^2} \ln\left(\frac{c}{b+c}\right) - \frac{\ell_\perp^2}{b+c} \right]\Bigg|_{\ell=\ell_{1,v}} . \tag{4.149}$$

Taking into account the condition $\alpha = (x-\beta)/\xi$ from the Dirac delta connecting DDs to GPDs (3.192), we have:

$$\begin{aligned} a = \ell^2 &= \left(\lambda_{12}\frac{x+\xi}{2\xi} - \lambda_1\right)^2 t - \left(\lambda_{12}\frac{x+\xi}{2\xi} - \lambda_1\right)\frac{\beta\lambda_{12}t}{\xi} + \bar{p}_\perp^2 \lambda_{12}^2 \beta^2 , \\ b = -2q'\ell &= Q^2 \left[\lambda_{12}\left(\frac{x+\xi}{2\xi} + \beta\frac{\rho-\xi}{2\xi^2}\frac{t}{Q^2}\right) - \lambda_1 + O(\text{tw-6}) \right] , \\ c &= Q'^2 + i0 , \\ \ell_\perp^2 &= \lambda_{12}^2 \beta^2 \bar{p}_\perp^2 , \end{aligned} \tag{4.150}$$

for operator $\mathscr{O}(\lambda_1,\lambda_2)$ and $\lambda_{12} = \lambda_1 - \lambda_2$. Formulas in (4.150) yield the following expressions for the functions integrated in $I_{(\mathrm{i})}$ and $I_{(\mathrm{ii})}$:

$$\frac{a}{b} = \frac{t}{2\xi Q^2}\left(\lambda_{12}(x+\xi) - 2\xi\lambda_1\right) - \frac{\beta\lambda_{12}t}{\xi Q^2} + \frac{\bar{p}_\perp^2}{Q^2}\frac{2\xi\lambda_{12}^2\beta^2}{\lambda_{12}(x+\xi)-2\xi\lambda_1} + O(\text{tw-6}) , \tag{4.151}$$

$$\frac{ac}{b^2}\ln\left(\frac{c}{b+c}\right) = -\left[\frac{t}{2\xi Q^2} - \frac{t}{Q^2\xi}\frac{\beta\lambda_{12}}{\lambda_{12}(x+\xi)-2\xi\lambda_1} + \frac{\bar{p}_\perp^2}{Q^2}\frac{2\xi\lambda_{12}^2\beta^2}{[\lambda_{12}(x+\xi)-2\xi\lambda_1]^2}\right]$$
$$\times(\xi-\rho)\ln\left(1+\frac{\lambda_{12}(x+\xi)-2\xi\lambda_1}{\xi-\rho+i0}\right) + O(\text{tw-6})\,, \quad (4.152)$$

$$\frac{\ell_\perp^2}{b+c} = \frac{\bar{p}_\perp^2}{Q^2}\frac{2\xi\lambda_{12}^2\beta^2}{\lambda_{12}(x+\xi)-2\xi\lambda_1+\xi-\rho+i0} + O(\text{tw-6})\,. \quad (4.153)$$

The next step would be to solve $I_{(0)}$. With the functional $\mathbb{I}_1$ introduced in App. K, we find:

$$I_{(0)} = -\frac{t}{Q^2}\frac{\xi-\rho}{\xi}\,\mathbb{I}_1\left[\frac{\xi}{x-\rho+i0}\right] = -\frac{t}{2Q^2}(\xi-\rho)\int_{-1}^{1}dx\,\partial_\xi\frac{H^{(+)}}{x-\rho+i0}\,, \quad (4.154)$$

where $\partial_\xi = \partial/\partial\xi$ is the derivative with respect to the skewness. The contribution by $I_{(0)}$ vanishes in the DVCS limit ($\rho\to\xi$), while it does not in TCS and DDVCS proving to be a genuine contribution to both of them.

The cases of $I_{(\mathrm{i})}$ and $I_{(\mathrm{ii})}$ are more involved due to the integration with respect to the auxiliary variables u and v. After solving these integrals and re-arranging terms, $I_{(\mathrm{i})}$ is:

$$I_{(\mathrm{i})} = \int_{-1}^{1}dx\iint_{\mathbb{D}}d\beta d\alpha\,2\Phi^{(+)}\delta(x-\beta-\alpha\xi)$$
$$\times\left[\frac{t}{Q^2}\left(\frac{x-\xi}{2\xi}-\frac{\beta}{\xi}\right)\mathbb{P}_{(\mathrm{i})} + \frac{\bar{p}_\perp^2\beta^2}{Q^2}\frac{2\xi}{x-\xi}\left(\mathbb{P}_{(\mathrm{i})}-\widetilde{\mathbb{P}}_{(\mathrm{i})}\right)\right]\,, \quad (4.155)$$

where we introduced the functions

$$\mathbb{P}_{(\mathrm{i})}(x/\xi,\rho/\xi) = 1 + \frac{\xi-\rho}{x-\xi}\mathrm{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right) \quad (4.156)$$

and

$$\widetilde{\mathbb{P}}_{(\mathrm{i})}(x/\xi,\rho/\xi) = 1 - \frac{\xi-\rho}{x-\xi}\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right)\,. \quad (4.157)$$

Making use of the functionals $\mathbb{I}_1$ and $\mathbb{I}_2$ from App. K, we can obtain an expression by means of the GPD $H^{(+)}$:

$$I_{(\mathrm{i})} = \frac{1}{2}\int_{-1}^{1}dx\,\left[\frac{t}{Q^2}\frac{x-\xi}{2\xi}\mathbb{P}_{(\mathrm{i})}\partial_x H^{(+)}\right] - \frac{t}{Q^2}\mathbb{I}_1\left[\mathbb{P}_{(\mathrm{i})}\right] + \frac{\bar{p}_\perp^2}{Q^2}\mathbb{I}_2\left[\frac{2\xi}{x-\xi}\left(\mathbb{P}_{(\mathrm{i})}-\widetilde{\mathbb{P}}_{(\mathrm{i})}\right)\right]$$
$$= \frac{1}{2}\int_{-1}^{1}dx\left\{-\frac{t}{Q^2}\left(\partial_x\frac{x-\xi}{2\xi}\mathbb{P}_{(\mathrm{i})}\right)H^{(+)} + \frac{t}{Q^2}\left[\frac{\mathbb{P}_{(\mathrm{i})}}{\xi}H^{(+)} - \partial_\xi\left(\mathbb{P}_{(\mathrm{i})}H^{(+)}\right)\right]\right.$$
$$\left. + \frac{\bar{p}_\perp^2}{Q^2}\xi^3\partial_\xi^2\frac{H^{(+)}}{\xi}\int_x^1 dx'\,\frac{2\xi}{x'-\xi}\left(\mathbb{P}_{(\mathrm{i})}(x'/\xi,\rho/\xi)-\widetilde{\mathbb{P}}_{(\mathrm{i})}(x'/\xi,\rho/\xi)\right)\right\}. \quad (4.158)$$

The integral with respect to x' is:

$$\int_x^1 dx'\,\frac{2\xi}{x'-\xi}\left(\mathbb{P}_{(\mathrm{i})}(x'/\xi,\rho/\xi)-\widetilde{\mathbb{P}}_{(\mathrm{i})}(x'/\xi,\rho/\xi)\right) =$$
$$= 2\xi\left(\mathbb{P}_{(\mathrm{i})}-\widetilde{\mathbb{P}}_{(\mathrm{i})}-\ln\left(\frac{x-\xi+i0}{x-\rho+i0}\right)\right) - (x\to 1)\,. \quad (4.159)$$

Removing the x-independent terms as the integrate to zero with the GPD $H^{(+)}$, we finally obtain

$$I_{(\text{i})} = \frac{1}{2}\int_{-1}^{1} dx \left\{ \frac{t}{Q^2}\left[\left(\frac{\mathbb{P}_{(\text{i})}}{\xi} - \frac{\widetilde{\mathbb{P}}_{(\text{i})}}{2\xi}\right) H^{(+)} - \partial_\xi\left(\mathbb{P}_{(\text{i})} H^{(+)}\right)\right] + \frac{\bar{p}_\perp^2}{Q^2} 2\xi^3 \partial_\xi^2 \left(\left[\mathbb{P}_{(\text{i})} - \widetilde{\mathbb{P}}_{(\text{i})} + \ln\left(\frac{x-\rho+i0}{x-\xi+i0}\right)\right] H^{(+)}\right)\right\}, \tag{4.160}$$

where $\mathbb{P}_{(\text{i})}$ and $\widetilde{\mathbb{P}}_{(\text{i})}$ have been redefined by removing the factors 1 in Eqs. (4.156) and (4.157) which integrate to zero. Now:

$$\mathbb{P}_{(\text{i})}(x/\xi, \rho/\xi) = \frac{\xi-\rho}{x-\xi}\text{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right) \tag{4.161}$$

and

$$\widetilde{\mathbb{P}}_{(\text{i})}(x/\xi, \rho/\xi) = -\frac{\xi-\rho}{x-\xi}\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right). \tag{4.162}$$

These functions vanish for the DVCS case ($\rho \to \xi$), but are finite for the TCS limit and non-singular at the pole $x = \xi$.

Alternatively, for $I_{(\text{ii})}$ we have:

$$I_{(\text{ii})} = \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi) \times \int_0^1 dv \left[\left(\frac{t}{Q^2}\frac{\bar{v}(x+\xi)-2\xi}{2\xi\bar{v}} - \frac{t}{Q^2}\frac{\beta}{\xi} + \frac{\bar{p}_\perp^2}{Q^2}\frac{2\xi\bar{v}\beta^2}{\bar{v}(x+\xi)-2\xi}\right) \times \left(1 - \frac{\xi-\rho}{\bar{v}(x+\xi)-2\xi}\ln\left(1+\frac{\bar{v}(x+\xi)-2\xi}{\xi-\rho+i0}\right)\right) - \frac{\bar{p}_\perp^2}{Q^2}\frac{2\xi\bar{v}\beta^2}{\bar{v}(x+\xi)-\xi-\rho+i0}\right]. \tag{4.163}$$

From here we identify the following integrals with respect to v:

$$I_{v,1} = \int_0^1 dv\, \frac{\bar{v}(x+\xi)-2\xi}{\bar{v}} = x + \xi + (x\text{-indep. terms}), \tag{4.164}$$

$$I_{v,2} = \int_0^1 dv\, \frac{\bar{v}}{\bar{v}(x+\xi)-2\xi} = \frac{1}{x+\xi} + \frac{2\xi}{(x+\xi)^2}\ln\left(\frac{x-\xi}{-2\xi}\right), \tag{4.165}$$

$$I_{v,3} = \int_0^1 dv\, \frac{\xi-\rho}{\bar{v}}\ln\left(1+\frac{\bar{v}(x+\xi)-2\xi}{\xi-\rho+i0}\right) = -(\xi-\rho)\text{Li}_2\left(-\frac{x+\xi}{-\xi-\rho+i0}\right) + (x\text{-indep. terms}), \tag{4.166}$$

$$I_{v,4} = \int_0^1 dv\, \frac{\xi-\rho}{\bar{v}(x+\xi)-2\xi}\ln\left(1+\frac{\bar{v}(x+\xi)-2\xi}{\xi-\rho+i0}\right) = -\frac{\xi-\rho}{x+\xi}\left[\text{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right) - (x\to-\xi)\right], \tag{4.167}$$

$$I_{v,5} = \int_0^1 dv\, \frac{(\xi-\rho)\bar{v}}{(\bar{v}(x+\xi)-2\xi)^2} \ln\left(1+\frac{\bar{v}(x+\xi)-2\xi}{\xi-\rho+i0}\right)$$
$$= -\frac{\xi-\rho}{(x+\xi)^2}\left[\frac{2\xi}{x-\xi}\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right)+\frac{2\xi}{\xi-\rho+i0}\ln\left(-\frac{x-\rho+i0}{x-\xi}\right)\right.$$
$$\left.+\,\mathrm{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right)-(x\to-\xi)\right], \tag{4.168}$$

$$I_{v,6} = \int_0^1 dv\, \frac{\bar{v}}{\bar{v}(x+\xi)-\xi-\rho+i0} = \frac{1}{x+\xi}+\frac{\xi+\rho}{(x+\xi)^2}\ln\left(\frac{x-\rho+i0}{-\xi-\rho+i0}\right), \tag{4.169}$$

which results in

$$I_{\text{(ii)}} = -\frac{1}{2}\int_{-1}^1 dx\, \frac{t}{Q^2}\left(\partial_x \frac{I_{v,1}-I_{v,3}}{\xi}\right)H^{(+)} + \frac{t}{Q^2}\mathbb{I}_1\left[I_{v,4}\right] + \frac{\bar{p}_\perp^2}{Q^2}\mathbb{I}_2\left[2\xi\left(I_{v,2}-I_{v,5}-I_{v,6}\right)\right]$$
$$= \frac{1}{2}\int_{-1}^1 dx\left\{\frac{t}{Q^2}\left[\left(\frac{\mathbb{P}_{\text{(ii)}}}{\xi}-\frac{\widetilde{\mathbb{P}}_{\text{(ii)}}}{2\xi}\right)H^{(+)}-\partial_\xi\left(\mathbb{P}_{\text{(ii)}}H^{(+)}\right)\right]+\frac{\bar{p}_\perp^2}{Q^2}2\xi^3\partial_\xi^2\left(\mathbb{P}_{\text{(ii)}}H^{(+)}\right)\right\}. \tag{4.170}$$

Here, we have defined

$$\mathbb{P}_{\text{(ii)}}(x/\xi,\rho/\xi) = \frac{\xi-\rho}{x+\xi}\left[\mathrm{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right)-(x\to-\xi)\right] \tag{4.171}$$

and

$$\widetilde{\mathbb{P}}_{\text{(ii)}}(x/\xi,\rho/\xi) = -\frac{\xi-\rho}{x+\xi}\ln\left(\frac{x-\rho+i0}{-\xi-\rho+i0}\right). \tag{4.172}$$

The coefficient functions $\mathbb{P}_{\text{(ii)}}$ and $\widetilde{\mathbb{P}}_{\text{(ii)}}$ vanish at the DVCS limit and are finite at the pole $x=-\xi$. Notice also that $\widetilde{\mathbb{P}}_{\text{(ii)}}$ is divergent as we approach the limit $\rho\to-\xi$. This is not a problem since this function will be compensated by contributions coming from other lines of the Compton tensor (3.163). To twist-4, these are:

$$\mathcal{A}^{++}\big|_{\text{tw-4}} \supset -\frac{g_{\perp,\mu\nu}}{2i\pi^2} i\int d^4z\, e^{iq'z}\left\{\frac{t}{2}\frac{z^\nu z^\mu}{(-z^2+i0)^2}\int_0^1 du\,\bar{u}\int_0^{\bar{u}} dv\, \mathscr{O}(\bar{u},v)\right.$$
$$+\frac{g^{\mu\nu}}{4(-z^2+i0)}\left[\int_0^1 du\int_0^{\bar{u}} dv\,\mathscr{O}_1(\bar{u},v)-\int_0^1 dv\,\mathscr{O}_2(1,v)\right]$$
$$\left.-\frac{z^\nu z^\mu}{(-z^2+i0)^2}\int_0^1 du\int_0^{\bar{u}} dv\,\left[(\ln\bar{\tau}+\ln\bar{u}+u)\mathscr{O}_1(\bar{u},v)+\left(\frac{v}{\bar{v}}+\bar{u}\right)\mathscr{O}_2(\bar{u},v)\right]\right\}. \tag{4.173}$$

Hence, we require the following Fourier transforms:

$$i\int d^4z\, e^{iq'z}\frac{tz^\nu z^\mu}{(-z^2+i0)^2}\left[e^{-i\ell z}\right]_{\text{LT}} = \frac{2\pi^2 t}{b+c}g^{\mu\nu}+(\text{terms}\sim q')+O(\text{tw-6}), \tag{4.174}$$

$$i\int d^4z\, e^{iq'z}\frac{1}{-z^2+i0}i\Delta\partial\left[e^{-i\ell z}\right]_{\rm LT} = -4\pi^2\frac{\Delta\ell}{b+c}+2\pi^2\frac{a}{b}\left\{\frac{Q^2}{b+c}+\frac{Q^2}{b}\ln\left(\frac{c}{b+c}\right)\right\} + O(\text{tw-6})\,. \tag{4.175}$$

$$i\int d^4z\, e^{iq'z}\frac{1}{-z^2+i0}\frac{t}{2}\left[e^{-i\ell z}\right]_{\rm LT} = -2\pi^2\frac{t}{b+c}+O(\text{tw-6})\,. \tag{4.176}$$

$$i\int d^4z\, e^{iq'z}\frac{z^\nu z^\mu}{(-z^2+i0)^2}i\Delta\partial\left[e^{-i\ell z}\right]_{\rm LT} = 2\pi^2 g^{\mu\nu}\frac{\Delta\ell}{b+c}-\pi^2 g^{\mu\nu}\frac{Q^2}{b}\left(\frac{a}{b+c}+\frac{a}{b}\ln\left(\frac{c}{b+c}\right)\right) + (\text{terms} \sim q',\Delta)+O(\text{tw-6})\,. \tag{4.177}$$

Making use of the expressions for a, b and c provided in Eqs. (4.150), we can write:

$$\frac{t}{b+c} = \frac{t}{Q^2}\frac{2\xi}{\lambda_{12}(x+\xi)-2\xi\lambda_1+\xi-\rho+i0}+O(\text{tw-6})\,, \tag{4.178}$$

$$\frac{\Delta\ell}{b+c} = \frac{t}{Q^2}\frac{\lambda_{12}(x+\xi)-2\xi\lambda_1-\lambda_{12}\beta}{\lambda_{12}(x+\xi)-2\xi\lambda_1+\xi-\rho+i0}+O(\text{tw-6})\,, \tag{4.179}$$

$$\frac{aQ^2}{b(b+c)} = \frac{t}{Q^2}\frac{\lambda_{12}(x+\xi)-2\xi\lambda_1-2\beta\lambda_{12}}{\lambda_{12}(x+\xi)-2\xi\lambda_1+\xi-\rho+i0} + \frac{\bar{p}_\perp^2}{Q^2}\frac{4\xi^2\lambda_{12}^2\beta^2}{[\lambda_{12}(x+\xi)-2\xi\lambda_1][\lambda_{12}(x+\xi)-2\xi\lambda_1+\xi-\rho+i0]}+O(\text{tw-6})\,, \tag{4.180}$$

$$\frac{aQ^2}{b^2}\ln\left(\frac{c}{b+c}\right) = -\left[\frac{t}{Q^2}\left(1-\frac{2\beta\lambda_{12}}{\lambda_{12}(x+\xi)-2\xi\lambda_1}\right)+\frac{\bar{p}_\perp^2}{Q^2}\frac{4\xi^2\lambda_{12}^2\beta^2}{(\lambda_{12}(x+\xi)-2\xi\lambda_1)^2}\right] \times\ln\left(1+\frac{\lambda_{12}(x+\xi)-2\xi\lambda_1}{\xi-\rho+i0}\right)+O(\text{tw-6})\,, \tag{4.181}$$

for operators $\mathscr{O}(\lambda_1,\lambda_2)$ with $\lambda_{12}=\lambda_1-\lambda_2$.

Gathering all these terms together and including the identity $1=\int_{-1}^{1}dx\,\delta(x-\beta-\alpha\xi)$ in Eq. (4.173), we get:

$$\mathcal{A}^{++}\big|_{\text{tw-4}} \supset I_{(\text{iii})}+I_{(\text{iv})}+I_{(\text{v})}+I_{(\text{vi})}\,, \tag{4.182}$$

where $I_{(\text{iii})}$ accounts for the first line of Eq. (4.173) and is given by

$$I_{(\text{iii})} = -\int_{-1}^{1}dx\iint_{\mathbb{D}}d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi)\int_0^1 du\,\bar{u}\int_0^{\bar{u}}dv\,\frac{1}{\bar{u}-v}\,\frac{t}{b+c}\bigg|_{\ell=\ell_{\bar{u},v}}\,. \tag{4.183}$$

It will cancel with a component of $I_{(\text{vi})}$. The integral $I_{(\text{iv})}$ collects the two terms in the second line of Eq. (4.173). After integrating with respect to u and v, as well as using the functionals in App. K to map the DD representation to the GPD one, it takes the form:

$$I_{(\text{iv})} = \int_{-1}^{1} dx \left[\frac{tH^{(+)}}{2Q^2} \frac{1}{x-\rho+i0} - \frac{t}{2Q^2} \left\{ \partial_\xi \left(\frac{-2\xi}{x+\xi} \left[\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0} \right) - \widetilde{\mathbb{P}}_{(\text{i})} \right] H^{(+)} \right. \right. \right.$$
$$\left. + \frac{\xi+\rho}{x+\xi} \ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0} \right) H^{(+)} \right) + \frac{2}{x+\xi} \left[\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0} \right) - \widetilde{\mathbb{P}}_{(\text{i})} \right] H^{(+)}$$
$$\left. \left. - \frac{\xi+\rho}{\xi} \frac{1}{x+\xi} \ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0} \right) H^{(+)} \right\} + \frac{\bar{p}_\perp^2}{Q^2} \xi^3 \partial_\xi^2 \left(\left[\widetilde{\mathbb{P}}_{(\text{i})} + \widetilde{\mathbb{P}}_{(\text{iii})} \right] H^{(+)} \right) \right], \tag{4.184}$$

where we defined the function

$$\widetilde{\mathbb{P}}_{(\text{iii})}(x/\xi, \rho/\xi) = -\frac{\xi+\rho}{x+\xi} \ln\left(\frac{x-\rho+i0}{-\xi-\rho+i0} \right). \tag{4.185}$$

This function is regular in the TCS, DVCS and $\rho \to -\xi$ limits. In the TCS case, $\rho \simeq -\xi(1-2t/Q'^2)$, the function $\widetilde{\mathbb{P}}_{(\text{iii})}$ contributes to twist-6. For $\rho \to -\xi$, it vanishes, while it has a finite limit at the pole $x = -\xi$.

The integral $I_{(\text{v})}$ accounts for the term that goes with $\mathscr{O}_1$ in last line of Eq. (4.173), while $I_{(\text{vi})}$ for that with $\mathscr{O}_2$. They are given by:

$$I_{(\text{v})} = \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha \, 2\Phi^{(+)} \delta(x-\beta-\alpha\xi)$$
$$\times \int_0^1 du \int_0^{\bar{u}} dv \, \frac{2}{\bar{u}-v} \left\{ \frac{t}{2Q^2} \frac{\bar{u}(x-\xi)-v(x+\xi)}{\bar{u}(x-\xi)-v(x+\xi)+\xi-\rho+i0} \right.$$
$$- \frac{\bar{p}_\perp^2}{Q^2} \frac{2\xi^2(\bar{u}-v)\beta^2}{[\bar{u}(x-\xi)-v(x+\xi)][\bar{u}(x-\xi)-v(x+\xi)+\xi-\rho+i0]}$$
$$+ \left[\frac{t}{2Q^2} \left(1 - \frac{2\beta(\bar{u}-v)}{\bar{u}(x-\xi)-v(x+\xi)} \right) + \frac{\bar{p}_\perp^2}{Q^2} \frac{2\xi^2(\bar{u}-v)^2\beta^2}{(\bar{u}(x-\xi)-v(x+\xi))^2} \right]$$
$$\left. \times \ln\left(1 + \frac{\bar{u}(x-\xi)-v(x+\xi)}{\xi-\rho+i0} \right) \right\} \left[\ln\left(\frac{\bar{u}-v}{1-v} \right) + \frac{1}{1-v} \right], \tag{4.186}$$

and

$$I_{(\text{vi})} = \int_{-1}^{1} dx \, \frac{tH^{(+)}}{Q^2} \left\{ \frac{2\xi}{-\xi-\rho+i0} \left[\frac{\widetilde{\mathbb{P}}_{(\text{ii})}(+1,\rho/\xi)}{x+\xi} - \frac{\widetilde{\mathbb{P}}_{(\text{ii})}}{\xi-\rho+i0} \right] \right.$$
$$\left. - \frac{2\xi\widetilde{\mathbb{P}}_{(\text{i})}}{(-\xi-\rho+i0)(x+\xi)} \right\} - I_{(\text{iii})}. \tag{4.187}$$

The contribution by $I_{(\text{v})}$ is more involved than the others due to the coefficient

$$C_{\bar{u},v} = \ln\left(\frac{\bar{u}-v}{1-v} \right) + \frac{1}{1-v}. \tag{4.188}$$

Rearranging terms and using the prescriptions detailed in App. K,

$$I_{(\mathrm{v})} = \int_{-1}^{1} dx \left\{ -\frac{t}{4Q^2}\widetilde{J}H^{(+)} + \frac{t}{2Q^2}\left[\partial_\xi\left(\frac{\xi L}{2}H^{(+)}\right) - \frac{L}{2}H^{(+)}\right] + \frac{\bar{p}_\perp^2}{Q^2}\xi^3\partial_\xi^2\frac{H^{(+)}}{\xi}\left[\mathcal{Y}_1 + \mathcal{Y}_2\right]\right\}, \tag{4.189}$$

where $\widetilde{J}$ comes from the derivative with respect to x of the terms without β,

$$\widetilde{J} = \int_0^1 du \int_0^{\bar{u}} dv\, 2\frac{\bar{u}(x-\xi) - v(x+\xi) + 2(\xi-\rho)}{(\bar{u}(x-\xi) - v(x+\xi) + \xi - \rho + i0)^2} C_{\bar{u},v}\,, \tag{4.190}$$

while L comes from the single β-term after the use of prescription (K.4), this is

$$L = \int_0^1 du \int_0^{\bar{u}} dv\, \frac{-4}{\bar{u}(x-\xi) - v(x+\xi)} \ln\left(1 + \frac{\bar{u}(x-\xi) - v(x+\xi)}{\xi - \rho + i0}\right) C_{\bar{u},v}\,. \tag{4.191}$$

Finally, using the prescription (K.5) for the β^2-terms, we get:

$$\mathcal{Y}_1 = \int_x^1 dx' \int_0^1 du \int_0^{\bar{u}} dv\, \frac{4\xi^2(\bar{u}-v)C_{\bar{u},v}}{(\bar{u}(x'-\xi) - v(x'+\xi))^2} \ln\left(1 + \frac{\bar{u}(x'-\xi) - v(x'+\xi)}{\xi - \rho + i0}\right), \tag{4.192}$$

$$\mathcal{Y}_2 = \int_x^1 dx' \int_0^1 du \int_0^{\bar{u}} dv\, \frac{-4\xi^2(\bar{u}-v)C_{\bar{u},v}}{[\bar{u}(x'-\xi) - v(x'+\xi)][\bar{u}(x'-\xi) - v(x'+\xi) + \xi - \rho + i0]}\,. \tag{4.193}$$

We can relate these two formulas to the L integral. To simplify the notation, let us rename:

$$Z(x') = -\frac{\bar{u}(x'-\xi) - v(x'+\xi)}{\xi - \rho + i0}\,, \tag{4.194}$$

so that

$$\begin{aligned}
\mathcal{Y}_1 &= \int_x^1 dx' \int_0^1 du \int_0^{\bar{u}} dv\, \frac{-4\xi^2}{\bar{u}-v}\left(\partial_{x'}^2 \ln Z(x')\right) \ln\left(1 - Z(x')\right) C_{\bar{u},v} \\
&= \int_x^1 dx' \int_0^1 du \int_0^{\bar{u}} dv\, \frac{-4\xi^2}{\bar{u}-v}\Big\{\partial_{x'}\left[\left(\partial_{x'} \ln Z(x')\right) \ln\left(1 - Z(x')\right)\right] \\
&\qquad - \left(\partial_{x'} \ln Z(x')\right) \partial_{x'} \ln\left(1 - Z(x')\right)\Big\} C_{\bar{u},v} \\
&= \int_0^1 du \int_0^{\bar{u}} dv\, \frac{4\xi^2 C_{\bar{u},v}}{\bar{u}-v}\left\{\left(\partial_x \ln Z(x)\right) \ln\left(1 - Z(x)\right) - \int_x^1 dx'\, \frac{\partial_{x'} Z(x')}{Z(x')}\frac{\partial_{x'} Z(x')}{1 - Z(x')}\right\} \\
&\quad - (x \to 1) \\
&= -\mathcal{Y}_2 - \xi^2 L - (x \to 1)\,.
\end{aligned} \tag{4.195}$$

The terms given by "$(x \to 1)$" vanish upon integration with the GPD, so we can remove them henceforth.

Therefore, we are left with calculating $\widetilde{J}$ and L. The latter is safe in both the TCS and DVCS limits thanks to

$$\int_0^1 du \int_0^1 dw \left[\ln\left(\frac{\bar{u}(1-w)}{1-\bar{u}w}\right) + \frac{1}{1-\bar{u}w} \right] = 0\,, \tag{4.196}$$

after the substitution $v = \bar{u}w$ in $C_{\bar{u},v}$. Because L is safe in all cases, we might as well write

$$L = \int_0^1 dw \, \frac{-4}{x-\xi-w(x+\xi)} \int_0^1 du \;\ln\left(\xi-\rho+i0+\bar{u}[x-\xi-w(x+\xi)]\right) C_{\bar{u},\bar{u}w}\,. \tag{4.197}$$

For DDVCS and TCS, this integral is a complicated expression including logarithms, dilogarithms and trilogarithms. Hence, we leave it as a quantity to be computed numerically for DDVCS and TCS. For the DVCS case, this integral acquires a simpler form (as $\rho = \xi$) and we can compute it analytically:

$$L_{\text{DVCS}} = \lim_{\rho\to\xi} L = \frac{4}{x-\xi}\left[\text{Li}_2\left(\frac{x+\xi}{2\xi}\right) - \text{Li}_2\left(1\right)\right]. \tag{4.198}$$

On the other hand, for the general case of DDVCS, $\widetilde{J}$ may be written as:

$$\begin{aligned}
\widetilde{J} = {} & \frac{2(\xi-\rho)}{-\xi-\rho+i0}\frac{1}{x-\xi}\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right) \\
& + \frac{2(x+\rho)}{(x-\xi)(x+\xi)}\Bigg[-\text{Li}_2\left(-\frac{x+\rho}{-\xi-\rho+i0}\right) - \text{Li}_2\left(\frac{2\xi}{x+\xi}\right) + \text{Li}_2\left(\frac{x+\rho}{-\xi-\rho+i0}\frac{-2\xi}{x+\xi}\right) \\
& \quad + \text{Li}_2\left(\frac{x-\rho}{\xi-\rho+i0}\right) + \text{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right) + \text{Li}_2\left(\frac{x-\rho}{x+\xi}\right) \\
& \quad - \text{Li}_2\left(\frac{\xi-\rho+i0}{x+\xi}\right) - \text{Li}_2\left(\frac{x+\rho}{x+\xi}\right) + \text{Li}_2\left(-\frac{-\xi-\rho+i0}{x+\xi}\right) \\
& \quad + \text{Li}_2\left(-\frac{x-\xi}{-x-\rho+i0}\right) + \frac{1}{2}\ln^2\left(\frac{-\xi-\rho+i0}{-x-\rho+i0}\right) \Bigg] \\
& + \frac{\xi-\rho}{\xi(x+\xi)}\Bigg[-\text{Li}_2\left(\frac{2\xi}{\xi-\rho+i0}\right) - \text{Li}_2\left(-\frac{2\xi}{-\xi-\rho+i0}\right) + 2\text{Li}_2\left(1\right) \\
& \quad - \text{Li}_2\left(\frac{x-\xi}{x+\xi}\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right) - \text{Li}_2\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right) + \text{Li}_2\left(\frac{x-\xi}{x+\xi}\frac{\xi-\rho+i0}{-\xi-\rho+i0}\right) \\
& \quad - \text{Li}_2\left(\frac{\xi-\rho+i0}{-\xi-\rho+i0}\right) - \ln\left(\frac{2\xi}{x+\xi}\right)\ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right) \\
& \quad - \ln\left(-\frac{2\xi}{x+\xi}\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right)\ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right) \\
& \quad - \text{Li}_2\left(\frac{2\xi}{x+\xi}\frac{x-\rho+i0}{\xi-\rho+i0}\right) + \text{Li}_2\left(\frac{2\xi}{x+\xi}\right) \Bigg] \\
& - \frac{\xi+\rho}{\xi(x-\xi)}\Bigg[\text{Li}_2\left(\frac{x-\xi}{x+\xi}\right) - \text{Li}_2\left(\frac{x-\xi}{x+\xi}\frac{\xi-\rho+i0}{-\xi-\rho+i0}\right)
\end{aligned}$$

$$+ \ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right)\ln\left(\frac{2\xi}{x+\xi}\right) - \ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right)\ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\frac{x-\xi}{x+\xi}\right)\Bigg]. \tag{4.199}$$

The term in the first line of $\widetilde{J}$ (4.199) will compensate the $\rho \to -\xi$ divergences from $\widetilde{\mathbb{P}}_{(\text{ii})}$ in Eq. (4.170) and the ones in $I_{(\text{vi})}$ that we will show later on.

From the expression above we can deduce its DVCS limit, which is zero,

$$\widetilde{J}_{\text{DVCS}} = \lim_{\rho\to\xi} \widetilde{J} = 0\,, \tag{4.200}$$

as well as the TCS limit for which we have:

$$\begin{aligned}\widetilde{J}_{\text{TCS}} &= \lim_{Q^2\to 0} \widetilde{J} \\ &= \lim_{Q^2\to 0} \frac{4\xi}{-\xi-\rho+i0}\frac{1}{x-\xi}\ln\left(\frac{x-\xi+i0}{2\xi}\right) \\ &\quad + \frac{2}{x+\xi}\Bigg[\text{Li}_2\left(\frac{x+\xi}{2\xi}\right) - \text{Li}_2\left(\frac{2\xi}{-x+\xi+i0}\right) + \text{Li}_2\left(1\right) - \frac{1}{2}\ln^2\left(\frac{2\xi}{-x+\xi+i0}\right) \\ &\quad + \ln\left(\frac{2\xi}{-x+\xi+i0}\right)\ln\left(-\frac{2\xi}{x+\xi}\right)\Bigg] + (\text{terms that contribute to twist-6 in } \mathcal{A}^{++})\,.\end{aligned} \tag{4.201}$$

Here, we employed the Abel-Rogers identity [122]:

$$\text{Li}_2\left(a\right) + \text{Li}_2\left(b\right) - \text{Li}_2\left(ab\right) = \text{Li}_2\left(\frac{a-ab}{1-ab}\right) + \text{Li}_2\left(\frac{b-ab}{1-ab}\right) + \ln\left(\frac{1-a}{1-ab}\right)\ln\left(\frac{1-b}{1-ab}\right). \tag{4.202}$$

The term in the first line of $\widetilde{J}_{\text{TCS}}$ (4.201) will compensate the $\rho \to -\xi$ divergences from $\widetilde{\mathbb{P}}_{(\text{ii})}$ in $I_{(\text{ii})}$ (4.170) and $I_{(\text{vi})}$ (4.187).

Finally, with the definitions of $\widetilde{J}$ and L, as well as the relation (4.195), $I_{(\text{v})}$ adopts the form

$$I_{(\text{v})} = \frac{1}{2}\int_{-1}^{1} dx \left\{ -\frac{t}{2Q^2}\left(\widetilde{J}+L\right)H^{(+)} + \frac{t}{2Q^2}\partial_\xi\left(\xi L H^{(+)}\right) - \frac{\bar{p}_\perp^2}{Q^2}\xi^3\partial_\xi^2\left(\xi L H^{(+)}\right)\right\}. \tag{4.203}$$

4.5 Final result for $\mathcal{A}^{++}$ and its DVCS and TCS limits

With the different terms found so far, $\mathcal{A}^{++}$ can be decomposed into a LT and a twist-4 contribution:

$$\mathcal{A}^{++} \sim O\left(\frac{1}{Q^0}\right) + O\left(\frac{1}{Q^2}\right) + \cdots \sim \text{tw-2} + \text{tw-4} + \cdots\,. \tag{4.204}$$

Above, the ellipses denote higher-twist terms starting at twist-6. Thus, the transverse-helicity conserving amplitude is given by the sum of the contributions in Eq. (4.144) for the LT and Eqs. (4.154), (4.160), (4.170), (4.183), (4.184), (4.203) and (4.187) for the twist-4 part:

$$
\begin{aligned}
\mathcal{A}^{++} &= \mathcal{A}^{++}\big|_{\mathrm{LT}} + I_{(0)} + I_{(\mathrm{i})} + I_{(\mathrm{ii})} + I_{(\mathrm{iii})} + I_{(\mathrm{iv})} + I_{(\mathrm{v})} + I_{(\mathrm{vi})} \\
&= \frac{1}{2}\int_{-1}^{1} dx \left\{ -\left(1 - \frac{t}{2\mathbb{Q}^2} + \frac{t(\xi-\rho)}{\mathbb{Q}^2}\partial_\xi\right)\frac{H^{(+)}}{x-\rho+i0} \right.\\
&+ \frac{t}{\xi\mathbb{Q}^2}\left[\mathbb{P}_{(\mathrm{i})} + \mathbb{P}_{(\mathrm{ii})} - \frac{\widetilde{\mathbb{P}}_{(\mathrm{i})} - \widetilde{\mathbb{P}}_{(\mathrm{iii})}}{2} - \frac{\xi(J+L)}{2}\right.\\
&\qquad \left. - \frac{\xi}{x+\xi}\left(\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right) - \frac{\xi+\rho}{2\xi}\ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right) - \widetilde{\mathbb{P}}_{(\mathrm{i})}\right)\right]H^{(+)} \\
&- \frac{t}{\mathbb{Q}^2}\partial_\xi\left[\left(\mathbb{P}_{(\mathrm{i})} + \mathbb{P}_{(\mathrm{ii})} - \frac{\xi L}{2} - \frac{\xi}{x+\xi}\left(\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right)\right.\right.\right.\\
&\qquad \left.\left.\left. - \frac{\xi+\rho}{2\xi}\ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right) - \widetilde{\mathbb{P}}_{(\mathrm{i})}\right)\right)H^{(+)}\right] \\
&\left. + \frac{\bar{p}_\perp^2}{\mathbb{Q}^2}2\xi^3\partial_\xi^2\left[\left(\mathbb{P}_{(\mathrm{i})} + \mathbb{P}_{(\mathrm{ii})} - \frac{\widetilde{\mathbb{P}}_{(\mathrm{i})} - \widetilde{\mathbb{P}}_{(\mathrm{iii})}}{2} - \frac{\xi L}{2} + \ln\left(\frac{x-\rho+i0}{x-\xi+i0}\right)\right)H^{(+)}\right]\right\} \\
&+ O(\text{tw-6})\,.
\end{aligned}
\tag{4.205}
$$

Here, integral L was introduced in Eq. (4.197) and J has been defined as the regular part of $\widetilde{J}$:

$$
J = \widetilde{J} - \frac{2(\xi-\rho)}{-\xi-\rho+i0}\frac{1}{x-\xi}\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right), \tag{4.206}
$$

with $\widetilde{J}$ given in Eq. (4.199). The functions $\mathbb{P}_{(\mathrm{i})}$, $\widetilde{\mathbb{P}}_{(\mathrm{i})}$, $\mathbb{P}_{(\mathrm{ii})}$ and $\widetilde{\mathbb{P}}_{(\mathrm{iii})}$ were introduced in Eqs. (4.161), (4.162), (4.171) and (4.185), respectively, but for convenience we gather them here below:

$$
\begin{aligned}
\mathbb{P}_{(\mathrm{i})}(x/\xi,\rho/\xi) &= \frac{\xi-\rho}{x-\xi}\mathrm{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right), \\
\widetilde{\mathbb{P}}_{(\mathrm{i})}(x/\xi,\rho/\xi) &= -\frac{\xi-\rho}{x-\xi}\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right), \\
\mathbb{P}_{(\mathrm{ii})}(x/\xi,\rho/\xi) &= \frac{\xi-\rho}{x+\xi}\left[\mathrm{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right) - (x\to-\xi)\right], \\
\widetilde{\mathbb{P}}_{(\mathrm{iii})}(x/\xi,\rho/\xi) &= -\frac{\xi+\rho}{x+\xi}\ln\left(\frac{x-\rho+i0}{-\xi-\rho+i0}\right).
\end{aligned}
\tag{4.207}
$$

Note that the terms with power correction $t/(\xi\mathbb{Q}^2)$ are finite in the limit $\xi\to0$ as the functions that follow are proportional to ξ or $\xi\pm\rho$. Also, Eq. (4.205) is finite for both DVCS and TCS limits.

In particular, for DVCS we have that $J_{\text{DVCS}} = \widetilde{J}_{\text{DVCS}} = 0$, cf. Eq. (4.200), and L_{DVCS} is considered in Eq. (4.198). With this results,

$$\begin{aligned}
\mathcal{A}^{++}_{\text{DVCS}} &= \lim_{\rho\to\xi} \mathcal{A}^{++} \\
&= \frac{1}{2}\int_{-1}^{1} dx \left\{ -\left(1 - \frac{t}{2\mathbb{Q}^2}\right) \frac{H^{(+)}}{x - \xi + i0} \right. \\
&\quad - \frac{2t}{\mathbb{Q}^2}\left[\frac{1}{x+\xi}\ln\left(\frac{x-\xi+i0}{-2\xi}\right) + \frac{1}{x-\xi}\left(\text{Li}_2\left(\frac{x+\xi}{2\xi}\right) - \text{Li}_2(1)\right)\right] H^{(+)} \\
&\quad + \frac{t}{\mathbb{Q}^2}\partial_\xi \left[\left(\frac{2\xi}{x-\xi}\left(\text{Li}_2\left(\frac{x+\xi}{2\xi}\right) - \text{Li}_2(1)\right) + \frac{\xi}{x+\xi}\ln\left(\frac{x-\xi+i0}{-2\xi}\right)\right) H^{(+)}\right] \\
&\quad \left. - \frac{\bar{p}_\perp^2}{\mathbb{Q}^2} 2\xi^3 \partial_\xi^2 \left[\left(\frac{2\xi}{x-\xi}\left(\text{Li}_2\left(\frac{x+\xi}{2\xi}\right) - \text{Li}_2(1)\right) + \frac{\xi}{x+\xi}\ln\left(\frac{x-\xi+i0}{-2\xi}\right)\right) H^{(+)}\right] \right\} \\
&\quad + O(\text{tw-6})\,, \qquad (4.208)
\end{aligned}$$

where the scale of DVCS is simply

$$\mathbb{Q}^2 = Q^2 + t\,. \qquad (4.209)$$

Equation (4.208) matches that of Ref. [47] taking into account that the GPD used there ($H^{(+)}_{[47]}$) relates to the one we employ here ($H^{(+)}$) via: $H^{(+)}_{[47]} = H^{(+)}/4$.
In turn, the TCS limit of Eq. (4.205) corresponds to

$$\begin{aligned}
\mathcal{A}^{++}_{\text{TCS}} &= \lim_{Q^2\to 0} \mathcal{A}^{++} \\
&= \frac{1}{2}\int_{-1}^{1} dx \left\{ -\left(1 - \frac{5}{2}\frac{t}{\mathbb{Q}^2}\right) \frac{H^{(+)}}{x + \xi(1 - 2t/Q'^2) + i0} \right. \\
&\quad + \frac{2t}{\mathbb{Q}^2}\left[\frac{1}{x-\xi}\text{Li}_2\left(-\frac{x-\xi}{2\xi}\right) + \frac{1}{x+\xi}\left(\text{Li}_2\left(-\frac{x-\xi}{2\xi}\right) - \text{Li}_2(1)\right)\right. \\
&\qquad \left. - \frac{\xi}{4}(L_{\text{TCS}} + J_{\text{TCS}})\right] H^{(+)} \\
&\quad - \frac{t}{\mathbb{Q}^2}\partial_\xi \left[\left(\frac{2\xi}{x+\xi+i0} + \frac{2\xi}{x-\xi}\text{Li}_2\left(-\frac{x-\xi}{2\xi}\right) + \frac{2\xi}{x+\xi}\left(\text{Li}_2\left(-\frac{x-\xi}{2\xi}\right) - \text{Li}_2(1)\right)\right.\right. \\
&\qquad \left.\left. - \frac{\xi L_{\text{TCS}}}{2} - \frac{\xi}{x-\xi}\ln\left(\frac{x+\xi+i0}{2\xi}\right)\right) H^{(+)}\right] \\
&\quad + \frac{\bar{p}_\perp^2}{\mathbb{Q}^2} 2\xi^3 \partial_\xi^2 \left[\left(\frac{2\xi}{x-\xi}\text{Li}_2\left(-\frac{x-\xi}{2\xi}\right) + \frac{2\xi}{x+\xi}\left(\text{Li}_2\left(-\frac{x-\xi}{2\xi}\right) - \text{Li}_2(1)\right)\right.\right. \\
&\qquad \left.\left.\left. - \frac{\xi L_{\text{TCS}}}{2} + \frac{\xi}{x-\xi}\ln\left(\frac{x+\xi+i0}{2\xi}\right) + \ln\left(\frac{x+\xi+i0}{x-\xi+i0}\right)\right) H^{(+)}\right]\right\} \\
&\quad + O(\text{tw-6})\,, \qquad (4.210)
\end{aligned}$$

with the scale of TCS given by

$$\mathbb{Q}^2 = Q'^2 + t\,. \tag{4.211}$$

The evaluation at $\mathbb{Q}^2 = 0$ renders a value of ρ that to the twist-4 accuracy can be approximated to $\rho \simeq -\xi(1-2t/Q'^2)$. The integral J_{TCS} is given by combining Eqs. (4.201) and (4.206), whilst L_{TCS} is obtained through L in Eq. (4.197) via

$$L_{\text{TCS}} = \lim_{\mathbb{Q}^2\to 0} L = \lim_{\rho\to -\xi(1-2t/Q'^2)} L\,. \tag{4.212}$$

Because to the accuracy of twist-4 in TCS we have $\rho \simeq -\xi(1-2t/Q'^2)$, it happens that the LT-like component of DDVCS does not produce the LT term of TCS with no t-dependence, but a hard-coefficient function of the form

$$-\frac{1}{x+\xi(1-2t/Q'^2)+i0}\,, \tag{4.213}$$

which represents a single pole at a point slightly perturbed away from that of the LT ($x = -\xi$). Of course, at LT accuracy one can take the approximation $\rho \simeq -\xi$ and recover the usual result:

$$\mathcal{A}^{++}_{\text{TCS}}\big|_{\text{LT}} = \lim_{Q'^2\to\infty} \mathcal{A}^{++}_{\text{TCS}} = -\frac{1}{2}\int_{-1}^{1} dx\, \frac{H^{(+)}}{x+\xi+i0}\,. \tag{4.214}$$

4.6 Transverse-helicity flip amplitude, $\mathcal{A}^{+-}$

In this section we describe the transverse-helicity flip amplitude, denotes as $\mathcal{A}^{+-}$. At LO, this amplitude appears as a kinematic higher-twist correction, whereas at NLO and LT accuracy it receives contributions from the gluon transversity GPD [123]. Since we work at zeroth order in α_s, the contribution detailed here will start at higher-twist, in particular at twist-4.

Due to the form of corresponding projector, $\Pi^{(+-)}_{\mu\nu}$ (4.102), terms proportional to the metric, to longitudinal vectors or to an antisymmetric tensor will vanish. Consequently, to twist-4 only the terms $\sim z^\mu \partial^\nu \mathscr{O}(\bar{u},0)$ and $\sim z^\nu \partial^\mu \mathscr{O}(1,v)$ in the first line of Compton tensor (3.163) contribute. After integration by parts:

$$\begin{aligned}
\mathcal{A}^{+-}\big|_{\text{tw-4}} &= \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, \Phi^{(+)}\delta(x-\beta-\alpha\xi) \\
&\quad \times \frac{1}{i\pi^2} i \int d^4z\, e^{iq'z}\, \Pi^{(+-)}_{\mu\nu} \frac{2z^\mu z^\nu}{(-z^2+i0)^3}\left[\int_0^1 du\, \mathscr{O}(\bar{u},0) + \int_0^1 dv\, \mathscr{O}(1,v)\right] \\
&= \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, \Phi^{(+)}\delta(x-\beta-\alpha\xi)\frac{\pi^2}{2} \\
&\quad \times \left(\int_0^1 du\, \frac{\Pi^{(+-)}_{\mu\nu}\ell^\mu\ell^\nu}{(q'-\ell)^2+i0}\bigg|_{\ell=\ell_{\bar{u},0}} + \int_0^1 dv\, \frac{\Pi^{(+-)}_{\mu\nu}\ell^\mu\ell^\nu}{(q'-\ell)^2+i0}\bigg|_{\ell=\ell_{1,v}}\right),
\end{aligned} \tag{4.215}$$

where for the light-ray operator $\mathscr{O}(\lambda_1, \lambda_2)$ we have

$$\ell^\mu = \ell^\mu_{\lambda_1,\lambda_2} = -\lambda_1 \Delta^\mu - \lambda_{12}\left[\beta \bar{p}^\mu - \frac{1}{2}(\alpha+1)\Delta^\mu\right], \tag{4.216}$$

hence

$$\Pi^{(+-)}_{\mu\nu} \ell^\mu \ell^\nu = -\lambda_{12}^2 \beta^2 \frac{\bar{p}_\perp^2}{2}, \quad \lambda_{12} = \lambda_1 - \lambda_2 . \tag{4.217}$$

Making use of the functional $\mathbb{I}_2$ in App. K, we finally obtain

$$\mathcal{A}^{+-} = \frac{\bar{p}_\perp^2}{\mathbb{Q}^2} 4(\xi^2 \partial_\xi)^2 \int_{-1}^{1} \frac{dx}{2\xi} \left(\widetilde{\mathbb{P}}_{\rm (iii)} - \widetilde{\mathbb{P}}_{\rm (i)} + 2 \ln\left(\frac{x-\rho+i0}{-2\xi}\right)\right) H^{(+)} + O(\text{tw-6}), \tag{4.218}$$

where functions $\widetilde{\mathbb{P}}_{\rm (i)}$ and $\widetilde{\mathbb{P}}_{\rm (iii)}$ are collected in Eqs. (4.207). The DVCS and TCS limits of $\mathcal{A}^{+-}$ up to twist-4 are given by

$$\mathcal{A}^{+-}_{\rm DVCS} = \frac{\bar{p}_\perp^2}{\mathbb{Q}^2} 4(\xi^2 \partial_\xi)^2 \int_{-1}^{1} \frac{dx}{2\xi} \frac{2x}{x+\xi} \ln\left(\frac{x-\xi+i0}{-2\xi}\right) H^{(+)} + O(\text{tw-6}), \tag{4.219}$$

$$\mathcal{A}^{+-}_{\rm TCS} = \frac{\bar{p}_\perp^2}{\mathbb{Q}^2} 4(\xi^2 \partial_\xi)^2 \int_{-1}^{1} \frac{dx}{2\xi} \frac{2x}{x-\xi} \ln\left(\frac{x+\xi+i0}{2\xi}\right) H^{(+)} + O(\text{tw-6}). \tag{4.220}$$

Our calculation of the DVCS amplitude above as a limiting case of DDVCS agrees with that the earlier publication [47]. Note that the two limits are related by

$$\mathcal{A}^{+-}_{\rm TCS} = \left(\mathcal{A}^{+-}_{\rm DVCS}\right)\Big|_{\substack{\xi\to-\xi \\ Q^2 \leftrightarrow -Q'^2}}, \tag{4.221}$$

where $Q^2 \leftrightarrow -Q'^2$ is a consequence of going from TCS (with a timelike virtuality) to DVCS (with a spacelike one), and $\xi \to -\xi$ is expected due to time reversal symmetry. This transformation would also change $\mathcal{A}^{+-}$ to $\mathcal{A}^{-+}$ but, thanks to parity conservation, they are equal. In Eq. (4.221), we took into account that $Q^2 \leftrightarrow -Q'^2$ implies $\mathbb{Q}^2 \to -\mathbb{Q}^2$ (up to a t factor that we can ignore as it produces a higher twist).

These expressions complete the calculation of the transverse-helicity flip amplitude.

4.7 Longitudinal-to-transverse and transverse-to-longitudinal helicity flip amplitudes, $\mathcal{A}^{0+}$ and $\mathcal{A}^{+0}$

For the cases of the helicity-flip amplitudes $\mathcal{A}^{0+}$ and $\mathcal{A}^{+0}$, the corresponding projectors were introduced earlier in Eqs. (4.99) and (4.100):

$$\Pi^{(0+)}_{\mu\nu} = \bar{p}_{\perp,\mu} q'_\nu \frac{Q}{\sqrt{2}|\bar{p}_\perp|R}, \tag{4.222}$$

$$\Pi^{(+0)}_{\mu\nu} = -q_\mu \bar{p}_{\perp,\nu} \frac{iQ'}{\sqrt{2}|\bar{p}_\perp|R}. \tag{4.223}$$

They satisfy

$$\Pi^{(0+,+0)}_{\mu\nu}\left(g^{\mu\nu}, n^\mu n'^\nu, n^\nu n'^\mu\right) = 0, \tag{4.224}$$

$$\Pi^{(0+)}_{\mu\nu}(q'^{\mu}\ell^{\nu}) = 0\,, \qquad \Pi^{(+0)}_{\mu\nu}(q'^{\mu}\ell^{\nu}) = \frac{-i(qq')}{\sqrt{2}R}Q'\frac{\ell\bar{p}_{\perp}}{|\bar{p}_{\perp}|}\,, \tag{4.225}$$

$$\Pi^{(0+)}_{\mu\nu}(q'^{\nu}\ell^{\mu}) = \frac{Q'^2}{\sqrt{2}R}Q\frac{\ell\bar{p}_{\perp}}{|\bar{p}_{\perp}|}\,, \qquad \Pi^{(+0)}_{\mu\nu}(q'^{\nu}\ell^{\mu}) = 0\,, \tag{4.226}$$

$$\Pi^{(0+)}_{\mu\nu}(\ell^{\mu}\ell^{\nu}) = \frac{\ell q'}{\sqrt{2}R}Q\frac{\ell\bar{p}_{\perp}}{|\bar{p}_{\perp}|}\,, \qquad \Pi^{(+0)}_{\mu\nu}(\ell^{\mu}\ell^{\nu}) = \frac{-i(q\ell)}{\sqrt{2}R}Q'\frac{\ell\bar{p}_{\perp}}{|\bar{p}_{\perp}|}\,, \tag{4.227}$$

$$\Pi^{(0+)}_{\mu\nu}(\Delta^{\nu}\ell^{\mu}) = -\frac{\mathbb{Q}^2}{2\sqrt{2}R}Q\frac{\ell\bar{p}_{\perp}}{|\bar{p}_{\perp}|}\,, \qquad \Pi^{(+0)}_{\mu\nu}(\Delta^{\mu}\ell^{\nu}) = \frac{-i(q\Delta)}{\sqrt{2}R}Q'\frac{\ell\bar{p}_{\perp}}{|\bar{p}_{\perp}|}\,. \tag{4.228}$$

To understand the kinematic power counting of the different terms in the Compton tensor, we must consider:

$$\frac{\mathbb{Q}^2}{2R} = 1 + O(\text{tw-4})\,,\ \text{see Eq. (4.136)}\,, \tag{4.229}$$

$$\ell\bar{p}_{\perp} \sim \bar{p}_{\perp}^2\,, \tag{4.230}$$

$$\ell q = \ell\Delta + \ell q' \sim At + \left(A'\frac{\mathbb{Q}^2}{2} + B'\frac{R}{2\xi}\right) \sim At + B\mathbb{Q}^2\,, \tag{4.231}$$

$$\ell q' \sim A'(\Delta q') + B'(\bar{p}q') = A'\frac{\mathbb{Q}^2}{2} + B'\frac{R}{2\xi} \sim \mathbb{Q}^2\,, \tag{4.232}$$

$$qq' = Q'^2 - \frac{\mathbb{Q}^2}{2}\,, \tag{4.233}$$

where A, B, A' and B' are order one quantities. Then, it is clear that the non-zero projections in Eqs. (4.225) to (4.228) scale as

$$\Pi^{(0+)}_{\mu\nu}\left(q'^{\nu}\ell^{\mu},\,\ell^{\mu}\ell^{\nu},\,\Delta^{\nu}\ell^{\mu}\right) \sim Q|\bar{p}_{\perp}|\,, \tag{4.234}$$

$$\Pi^{(+0)}_{\mu\nu}\left(q'^{\mu}\ell^{\nu},\,\ell^{\mu}\ell^{\nu},\,\Delta^{\mu}\ell^{\nu}\right) \sim Q'|\bar{p}_{\perp}|\,. \tag{4.235}$$

As a consequence, they generate twist-3 contributions whenever they encounter a factor that scales as $1/\mathbb{Q}^2$. In the context of the integrals $I_{n,m}$ (4.111), $1/\mathbb{Q}^2$ factors appear through the terms $1/b$ and $1/(b+c)$. Therefore, the kinematic power expansion of $\mathcal{A}^{0+}$ and $\mathcal{A}^{+0}$ is:

$$\mathcal{A}^{0+,+0} \sim O\left(\frac{1}{\mathbb{Q}}\right) + \cdots \sim \text{tw-3} + \cdots\,, \tag{4.236}$$

where the ellipsis stand for the higher-twist terms starting at twist-5. The twist-3 contribution comes from the first and second lines of the OPE (3.163):

$$\begin{aligned}\mathcal{A}^{0+,+0}\big|_{\text{tw-3}} \supset\ & -\frac{1}{i\pi^2}i\int d^4z\, e^{iq'z}\,\Pi^{(0+,+0)}_{\mu\nu}\\ &\times\left\{\frac{1}{(-z^2+i0)^2}\left[z^{\mu}\partial^{\nu}\int_0^1 du\ \mathscr{O}(\bar{u},0) + z^{\nu}(\partial^{\mu}-i\Delta^{\mu})\int_0^1 dv\ \mathscr{O}(1,v)\right]\right.\\ &\left.+\frac{i}{2(-z^2+i0)}(\Delta^{\nu}\partial^{\mu}-\Delta^{\mu}\partial^{\nu})\int_0^1 du\int_0^{\bar{u}} dv\ \mathscr{O}(\bar{u},v)\right\}.\end{aligned} \tag{4.237}$$

The required Fourier transforms are:

$$\Pi^{(0+)}_{\mu\nu} i \int d^4z\, e^{iq'z} \frac{z^\mu \partial^\nu}{(-z^2+i0)^2} \left[e^{-i\ell z}\right]_{\rm LT} = -2\pi^2 \frac{\Pi^{(0+)}_{\mu\nu} \ell^\mu \ell^\nu}{b+c} + O(\text{tw-5})\,, \tag{4.238}$$

$$\Pi^{(+0)}_{\mu\nu} i \int d^4z\, e^{iq'z} \frac{z^\mu \partial^\nu}{(-z^2+i0)^2} \left[e^{-i\ell z}\right]_{\rm LT} = 2\pi^2 \frac{\Pi^{(+0)}_{\mu\nu} (q'^\mu \ell^\nu - \ell^\mu \ell^\nu)}{b+c} + O(\text{tw-5})\,, \tag{4.239}$$

$$\Pi^{(0+)}_{\mu\nu} i \int d^4z\, e^{iq'z} \frac{z^\nu (\partial^\mu - i\Delta^\mu)}{(-z^2+i0)^2} \left[e^{-i\ell z}\right]_{\rm LT} = 2\pi^2 \frac{\Pi^{(0+)}_{\mu\nu} (q'^\nu \ell^\mu - \ell^\nu \ell^\mu)}{b+c} + O(\text{tw-5})\,, \tag{4.240}$$

$$\Pi^{(+0)}_{\mu\nu} i \int d^4z\, e^{iq'z} \frac{z^\nu (\partial^\mu - i\Delta^\mu)}{(-z^2+i0)^2} \left[e^{-i\ell z}\right]_{\rm LT} = -2\pi^2 \frac{\Pi^{(+0)}_{\mu\nu} (\Delta^\mu \ell^\nu + \ell^\mu \ell^\nu)}{b+c} + O(\text{tw-5})\,, \tag{4.241}$$

$$\Pi^{(0+)}_{[\mu\nu]} i \int d^4z\, e^{iq'z} \frac{\Delta^\mu \partial^\nu}{-z^2+i0} \left[e^{-i\ell z}\right]_{\rm LT} = \frac{2\pi^2 i}{\sqrt{2}} \frac{(\ell \bar{p}_\perp) Q}{|\bar{p}_\perp|(b+c)} + O(\text{tw-5})\,, \tag{4.242}$$

$$\Pi^{(+0)}_{[\mu\nu]} i \int d^4z\, e^{iq'z} \frac{\Delta^\mu \partial^\nu}{-z^2+i0} \left[e^{-i\ell z}\right]_{\rm LT} = -\frac{2\pi^2}{\sqrt{2}} \frac{(\ell \bar{p}_\perp) Q'}{|\bar{p}_\perp|(b+c)} + O(\text{tw-5})\,. \tag{4.243}$$

As usual, antisymmetrization is given by $\Pi^{(AB)}_{[\mu\nu]} = (\Pi^{(AB)}_{\mu\nu} - \Pi^{(AB)}_{\nu\mu})/2$.

Including the identity $1 = \int_{-1}^{1} dx\, \delta(x-\beta-\alpha\xi)$ which relates DDs to GPDs, as well as the expressions for a, b, c and $\ell_\perp$ in (4.150), we obtain:

$$\begin{aligned}
\mathcal{A}^{+0} = \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 4\Phi^{(+)} \delta(x-\beta-\alpha\xi) \frac{i\beta Q' |\bar{p}_\perp|}{\sqrt{2} Q^2} \\
\int_0^1 du\, \frac{\bar{u}(x-\xi) - 2\rho}{\bar{u}(x-\xi)+\xi-\rho+i0} + \int_0^1 dv\, \frac{\bar{v}(x+\xi)}{\bar{v}(x+\xi)-\xi-\rho+i0} \\
- \int_0^1 du \int_0^{\bar{u}} dv\, \frac{2\xi}{\bar{u}(x-\xi) - v(x+\xi) + \xi - \rho + i0}
\end{aligned} \tag{4.244}$$

and

$$\begin{aligned}
\mathcal{A}^{0+} = \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 4\Phi^{(+)} \delta(x-\beta-\alpha\xi) \frac{-\beta Q |\bar{p}_\perp|}{\sqrt{2} Q^2} \\
\times \Bigg[\int_0^1 du\, \frac{\bar{u}(x-\xi)}{\bar{u}(x-\xi)+\xi-\rho+i0} + \int_0^1 dv\, \frac{\bar{v}(x+\xi) - 2\rho}{\bar{v}(x+\xi)-\xi-\rho+i0} \\
+ \int_0^1 du \int_0^{\bar{u}} dv\, \frac{2\xi}{\bar{u}(x-\xi) - v(x+\xi) + \xi - \rho + i0} \Bigg] + O(\text{tw-5})\,.
\end{aligned} \tag{4.245}$$

Solving the integrals with respect to u and v, and making use of the prescriptions described in App. K:

$$\mathcal{A}^{+0} = \frac{-iQ'|\bar{p}_\perp|}{\sqrt{2} Q^2} \int_{-1}^{1} dx\, 2\xi^2 \partial_\xi \left(\frac{1}{x-\xi} \ln\left(\frac{x-\rho+i0}{\xi-\rho+i0} \right) H^{(+)} \right) + O(\text{tw-5})\,, \tag{4.246}$$

$$\mathcal{A}^{0+} = \frac{-Q|\bar{p}_\perp|}{\sqrt{2} Q^2} \int_{-1}^{1} dx\, 2\xi^2 \partial_\xi \left(\frac{1}{x+\xi} \ln\left(\frac{x-\rho+i0}{-\xi-\rho+i0} \right) H^{(+)} \right) + O(\text{tw-5})\,. \tag{4.247}$$

Notice that $\mathcal{A}^{+0}$ ($\mathcal{A}^{0+}$) vanishes as we approach the DVCS (TCS) limit. This is expected since the outgoing (incoming) photon for DVCS (TCS) is real, hence it is transversely polarized.

However, as $\rho \to -\xi$ the amplitude $\mathcal{A}^{0+}$ is logarithmically divergent. The point $\rho = -\xi$ is physical for a timelike-dominated DDVCS ($Q'^2 > Q^2$) as it only imposes $t = -Q^2$ and for the twist expansion to hold $Q^2 < \mathbb{Q}^2 \simeq Q'^2$.

This divergence originates from the integral with operator $\mathscr{O}(1,v)$ in Eq. (4.237). The complete term producing the twist-3 contribution from $\mathscr{O}(1,v)$, prior to expanding by twists, is:

$$\begin{aligned}\mathcal{I} &= \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 4\Phi^{(+)}\delta(x-\beta-\alpha\xi)\frac{Q}{\sqrt{2}R|\bar{p}_\perp|}\int_0^1 d\bar{v}\, \frac{[(\ell q')-Q'^2](\ell\bar{p}_\perp)}{\bar{v}(a+b+c)}\bigg|_{\ell=\ell_{1,v}} \\ &= \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 4\Phi^{(+)}\delta(x-\beta-\alpha\xi)\frac{1}{\sqrt{2}} \\ &\quad\times \int_0^1 d\bar{v}\, \left[\frac{\mathbb{Q}^2\rho}{2R\xi} - \bar{v}\left(\frac{\beta}{2\xi}\left(1-\frac{\mathbb{Q}^2}{2R}\right)+\frac{\mathbb{Q}^2(x+\xi)}{4R\xi}\right)\right]\frac{\beta|\bar{p}_\perp|Q}{\ell_{1,v}^2+\mathbb{Q}^2\mathbb{F}},\end{aligned} \tag{4.248}$$

where

$$\mathbb{Q}^2\mathbb{F} = 2R\left[\frac{\bar{v}\beta}{2\xi}\left(1-\frac{\mathbb{Q}^2}{2R}\right)+\bar{v}\frac{\mathbb{Q}^2(x+\xi)}{4R\xi}-\frac{\mathbb{Q}^2(\xi+\rho-i0)}{4R\xi}\right]. \tag{4.249}$$

Thus far, the approximation employed to solve this integral consisted of neglecting powers higher that $|t|/\mathbb{Q}^2$ and $M^2/\mathbb{Q}^2$ by removing $\ell_{1,v}^2$ from the denominator in Eq. (4.248). This approximation triggers the logarithmic divergence discussed above, which is a consequence of the non-commutativity of the integration in $\mathcal{I}$ with respect to the auxiliary variable $\bar{v}$ and the twist expansion to the accuracy kept so far (twist-4). Therefore, we need to add higher-order terms. Since $\bar{v} \in (0,1)$ and

$$\ell_{1,v}^2 = t - 2\bar{v}\frac{x+\xi}{2\xi}t + \beta\frac{\bar{v}t}{\xi} + O(\bar{v}^2), \tag{4.250}$$

the next order in the approximation consists of keeping terms of the order of $|t|/\mathbb{Q}^2$ and $\bar{v}|t|/\mathbb{Q}^2$, but dropping those proportional to $\bar{v}^2|t|/\mathbb{Q}^2$ as they can be disregarded when compared to $\bar{v}|t|/\mathbb{Q}^2$ for $\bar{v} \in (0,1)$. Because $|\beta| \in (0,1)$, we also consider $\beta\bar{v}$ of the same order as $\bar{v}^2$ and neglect such product with respect to $\bar{v}$.

With this new approach, $\mathcal{I}$ takes the form:

$$\begin{aligned}\mathcal{I} &= \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 4\Phi^{(+)}\delta(x-\beta-\alpha\xi)\frac{1}{\sqrt{2}} \\ &\quad\times \frac{-\beta|\bar{p}_\perp|Q}{\mathbb{Q}^2}\left[\frac{\xi-\rho}{x+\xi}\ln\left(\frac{x(1-2t/\mathbb{Q}^2)-\rho+i0}{-\xi(1-2t/\mathbb{Q}^2)-\rho+i0}\right)+1\right] + O(\text{tw-7}).\end{aligned} \tag{4.251}$$

For the other contributions in Eq. (4.237) which come from operators $\mathscr{O}(\bar{u},0)$ and $\mathscr{O}(\bar{u},v)$, the new approach renders the same results. The reason for this is that $\ell_{\bar{u},0}^2$ is directly proportional to $\bar{u}^2$, while $\ell_{\bar{u},v}$ is proportional to $\bar{u}^2$, v^2 and $\bar{u}v$.

Introducing the expression (4.251) back in Eq. (4.237) and using the mapping between DDs and GPDs given in App. K, we get

$$\mathcal{A}^{0+} = \frac{-Q|\bar{p}_\perp|}{\sqrt{2}\mathbb{Q}^2}\int_{-1}^{1} dx\, 2\xi^2\partial_\xi\left(\frac{H^{(+)}}{2\xi}\left[\frac{\xi+\rho}{x+\xi}\ln\left(\frac{x-\rho+i0}{-\xi-\rho+i0}\right)\right.\right.$$
$$\left.\left.+\frac{\xi-\rho}{x+\xi}\ln\left(\frac{x(1-2t/\mathbb{Q}^2)-\rho+i0}{-\xi(1-2t/\mathbb{Q}^2)-\rho+i0}\right)\right]\right) + O(\text{tw-5})\,. \tag{4.252}$$

This new expression of $\mathcal{A}^{0+}$ is regular in the whole phase-space of DDVCS: The logarithmic divergence happens now for

$$\rho \to -\xi(1-2t/\mathbb{Q}^2)\,, \tag{4.253}$$

which is equivalent to $Q^2 \to 0$ (TCS) and is regulated by the global factor Q above. Indeed, we might as well write:

$$\mathcal{A}^{0+} = \frac{-Q|\bar{p}_\perp|}{\sqrt{2}\mathbb{Q}^2}\int_{-1}^{1} dx\, 2\xi^2\partial_\xi\left(\frac{H^{(+)}}{2\xi}\left[\frac{\xi+\rho}{x+\xi}\ln\left(\frac{x-\rho+i0}{-\xi-\rho+i0}\right)\right.\right.$$
$$\left.\left.+\frac{\xi-\rho}{x+\xi}\ln\left(\frac{x(1-2t/\mathbb{Q}^2)-\rho+i0}{-2\xi Q^2/\mathbb{Q}^2+i0}\right)\right]\right) + O(\text{tw-5})\,. \tag{4.254}$$

The TCS limit of this formula is trivially zero, while the DVCS one matches the result in Ref. [47] and is given by:

$$\mathcal{A}^{0+}_{\text{DVCS}} = \frac{-Q|\bar{p}_\perp|}{\sqrt{2}\mathbb{Q}^2}\int_{-1}^{1} dx\, 2\xi^2\partial_\xi\left(\frac{1}{x+\xi}\ln\left(\frac{x-\xi+i0}{-2\xi}\right)H^{(+)}\right) + O(\text{tw-5})\,. \tag{4.255}$$

In a similar manner, the DVCS limit of $\mathcal{A}^{+0}$ is zero whereas

$$\mathcal{A}^{+0}_{\text{TCS}} = \frac{-iQ'|\bar{p}_\perp|}{\sqrt{2}\mathbb{Q}^2}\int_{-1}^{1} dx\, 2\xi^2\partial_\xi\left(\frac{1}{x-\xi}\ln\left(\frac{x+\xi+i0}{2\xi}\right)\right) + O(\text{tw-5})\,. \tag{4.256}$$

Note that, as before, the two limits are related by

$$\mathcal{A}^{+0}_{\text{TCS}} = (\mathcal{A}^{0+}_{\text{DVCS}})\big|_{\substack{\xi\to-\xi \\ Q\to iQ'}}\,, \tag{4.257}$$

where $Q \to iQ'$ ensures $Q^2 \leftrightarrow -Q'^2$ as TCS has a timelike virtuality while DVCS a spacelike one, and $\xi \to -\xi$ is dictated by time reversal symmetry. Here, we took into account that $Q \to iQ'$ implies $\mathbb{Q}^2 \to -\mathbb{Q}^2$ (up to a t factor that we can ignore as it produces a higher twist).

4.8 Longitudinal-helicity conserving amplitude, $\mathcal{A}^{00}$

The projector onto $\mathcal{A}^{00}$, vid. Eq. (4.103), is an antisymmetric tensor built out of the longitudinal vectors q and q'. Hence, when contracted with the projector, transverse momenta and symmetric tensors vanish. Therefore, the longitudinal-helicity conserving amplitude takes the form:

$$\begin{aligned}\mathcal{A}^{00} =&\Pi^{(00)}_{\mu\nu} T^{\mu\nu} = \Pi^{(00)}_{\mu\nu} \frac{1}{i\pi^2} i \int d^4z\, e^{iq'z} \\ &\times \left\{ \frac{-1}{(-z^2+i0)^2} \left[z^\mu \partial^\nu \int_0^1 du\, \mathscr{O}(\bar{u},0) + z^\nu (\partial^\mu - i\Delta^\mu) \int_0^1 dv\, \mathscr{O}(1,v) \right] \right. \\ &- \frac{1}{-z^2+i0} \left[i\Delta^\nu \partial^\mu \int_0^1 du \int_0^{\bar{u}} dv\, \mathscr{O}(\bar{u},v) - \frac{t}{4} z^\mu \partial^\nu \int_0^1 du\, u \int_0^{\bar{u}} dv\, \mathscr{O}(\bar{u},v) \right] \\ &- \frac{iz^\mu \Delta^\nu}{4(-z^2+i0)} \int_0^1 du \int_0^{\bar{u}} dv \left[C_{(1),z\Delta}(\bar{u},v) \mathscr{O}_1(\bar{u},v) + C_{(2),z\Delta}(\bar{u},v) \mathscr{O}_2(\bar{u},v) \right] \\ &\left. + \frac{z^\nu \partial^\mu}{-z^2+i0} \int_0^1 du \int_0^{\bar{u}} dv \left[C_{(1),z\partial}(\bar{u},v) \mathscr{O}_1(\bar{u},v) + C_{(2),z\partial}(\bar{u},v) \mathscr{O}_2(\bar{u},v) \right] \right\}. \end{aligned} \tag{4.258}$$

where we defined the following conformal weights:

$$\begin{aligned} C_{(1),z\Delta}(\bar{u},v) =&\ \ln\left(\frac{\bar{u}-v}{\bar{u}(1-v)} \right) + 2\frac{(1-\bar{u})v}{\bar{u}-v} + 2\frac{v}{1-v} \left(1 + \frac{2(1-\bar{u})v}{\bar{u}-v} \right), \\ C_{(2),z\Delta}(\bar{u},v) =&\ \frac{v}{1-v} - 2\frac{\bar{u}v}{\bar{u}-v} - \left(\frac{v}{1-v} \right)^2 - \frac{v}{1-v}\delta(1-\bar{u}), \\ C_{(1),z\partial}(\bar{u},v) =& -\frac{(1-\bar{u})v}{\bar{u}-v} - \frac{1}{2} \left[\left(1 + \frac{2(1-\bar{u})v}{\bar{u}-v} \right) \frac{v}{1-v} + \ln \bar{u} + 1 - \bar{u} \right], \\ C_{(2),z\partial}(\bar{u},v) =&\ \frac{v}{\bar{u}-v} + \frac{1}{4} \left(\frac{v}{1-v} \right)^2 + \frac{1}{4} \frac{v}{1-v} \delta(1-\bar{u}) - \frac{1}{2}\bar{u} + \frac{1}{4}. \end{aligned} \tag{4.259}$$

Fourier transforms in the first line of Eq. (4.258) carry terms proportional to

$$\Pi^{(00)}_{\mu\nu} q'^\nu \ell^\mu = \underbrace{\frac{-i\mathrm{Q}^2(\ell q')}{R^2}}_{O(1)} QQ' - \underbrace{\frac{i2QQ'^3}{R^2}}_{O(1)} (\ell\Delta) \tag{4.260}$$

and

$$\Pi^{(00)}_{\mu\nu} q^\mu \ell^\nu = \underbrace{\frac{iQQ'\mathrm{Q}^2}{R^2}}_{O(1)} \left[(\ell q') \left(1 - \frac{2t}{\mathrm{Q}^2} \right) - (\Delta\ell)\frac{\rho}{\xi} \right]. \tag{4.261}$$

The former projection scales as QQ', while the latter as $\ell q' = O(\mathrm{Q}^2)$. Consequently, if the corresponding Fourier transform produces factors proportional to $1/\mathrm{Q}^2$, then such terms would be classified as leading-twist components. Each integral in the first line of Eq. (4.258) produces, apart from twist-4 components, LT terms which ultimately combine to give an x-independent contribution. This one nullifies when integrated with the GPD:

$$
\begin{aligned}
\mathcal{A}^{00}\big|_{\mathrm{LT}} = & \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 4\Phi^{(+)}\delta(x-\beta-\alpha\xi) \\
& \times \frac{iQQ'}{\mathbb{Q}^2}\left[\frac{-1}{\bar{u}}\frac{\xi-\rho}{\bar{u}(x-\xi)+\xi-\rho+i0} - (\xi\to-\xi,\,\rho\to\rho)\right] \\
= & \frac{iQQ'}{\mathbb{Q}^2}\int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 4\Phi^{(+)}\delta(x-\beta-\alpha\xi)\ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right) \\
= & -\frac{iQQ'}{\mathbb{Q}^2}\int_{-1}^{1} dx\, \partial_x\left(\ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right)\right) H^{(+)} \\
= & \, 0\,.
\end{aligned}
\tag{4.262}
$$

To organize the calculation of $\mathcal{A}^{00}$, we may decompose it as:

$$
\begin{aligned}
\mathcal{A}^{00} = & \mathcal{A}^{00}_{(0),z\partial} + \mathcal{A}^{00}_{(0),z(\partial-i\Delta)} + \mathcal{A}^{00}_{(0),\Delta\partial} + \mathcal{A}^{00}_{(0),tz\partial} \\
& + \mathcal{A}^{00}_{(1),z\Delta} + \mathcal{A}^{00}_{(2),z\Delta} + \mathcal{A}^{00}_{(1),z\partial} + \mathcal{A}^{00}_{(2),z\partial}\,,
\end{aligned}
\tag{4.263}
$$

where

$$
\mathcal{A}^{00}_{(0),z\partial} = \Pi^{(00)}_{\mu\nu}\frac{1}{i\pi^2}i\int d^4z\, e^{iq'z}\frac{-1}{(-z^2+i0)^2}z^\mu\partial^\nu\int_0^1 du\, \mathscr{O}(\bar{u},0)\,, \tag{4.264}
$$

$$
\mathcal{A}^{00}_{(0),z(\partial-i\Delta)} = \Pi^{(00)}_{\mu\nu}\frac{1}{i\pi^2}i\int d^4z\, e^{iq'z}\frac{-1}{(-z^2+i0)^2}z^\nu(\partial^\mu-i\Delta^\mu)\int_0^1 dv\, \mathscr{O}(1,v)\,, \tag{4.265}
$$

$$
\mathcal{A}^{00}_{(0),\Delta\partial} = \Pi^{(00)}_{\mu\nu}\frac{1}{i\pi^2}i\int d^4z\, e^{iq'z}\frac{-1}{-z^2+i0}i\Delta^\nu\partial^\mu\int_0^1 du\int_0^{\bar{u}} dv\, \mathscr{O}(\bar{u},v)\,, \tag{4.266}
$$

$$
\mathcal{A}^{00}_{(0),tz\partial} = \Pi^{(00)}_{\mu\nu}\frac{1}{i\pi^2}i\int d^4z\, e^{iq'z}\frac{1}{-z^2+i0}\frac{t}{4}z^\mu\partial^\nu\int_0^1 du\, u\int_0^{\bar{u}} dv\, \mathscr{O}(\bar{u},v)\,, \tag{4.267}
$$

$$
\mathcal{A}^{00}_{(1),z\Delta} = \Pi^{(00)}_{\mu\nu}\frac{1}{i\pi^2}i\int d^4z\, e^{iq'z}\frac{-iz^\mu\Delta^\nu}{4(-z^2+i0)}\int_0^1 du\int_0^{\bar{u}} dv\, C_{(1),z\Delta}(\bar{u},v)\mathscr{O}_1(\bar{u},v)\,, \tag{4.268}
$$

$$
\mathcal{A}^{00}_{(2),z\Delta} = \Pi^{(00)}_{\mu\nu}\frac{1}{i\pi^2}i\int d^4z\, e^{iq'z}\frac{-iz^\mu\Delta^\nu}{4(-z^2+i0)}\int_0^1 du\int_0^{\bar{u}} dv\, C_{(2),z\Delta}(\bar{u},v)\mathscr{O}_2(\bar{u},v)\,, \tag{4.269}
$$

$$
\mathcal{A}^{00}_{(1),z\partial} = \Pi^{(00)}_{\mu\nu}\frac{1}{i\pi^2}i\int d^4z\, e^{iq'z}\frac{z^\nu\partial^\mu}{-z^2+i0}\int_0^1 du\int_0^{\bar{u}} dv\, C_{(1),z\partial}(\bar{u},v)\mathscr{O}_1(\bar{u},v)\,, \tag{4.270}
$$

$$
\mathcal{A}^{00}_{(2),z\partial} = \Pi^{(00)}_{\mu\nu}\frac{1}{i\pi^2}i\int d^4z\, e^{iq'z}\frac{z^\nu\partial^\mu}{-z^2+i0}\int_0^1 du\int_0^{\bar{u}} dv\, C_{(2),z\partial}(\bar{u},v)\mathscr{O}_2(\bar{u},v)\,. \tag{4.271}
$$

By making use of the prescriptions detailed in App. K, each of these subamplitudes can be written (up to twist-4) as:

$$\begin{aligned}\mathcal{A}^{00}_{(0),z\partial} &= \frac{iQQ'}{\mathbb{Q}^2}\int_{-1}^{1}dx\int_0^1 du\,\frac{\xi-\rho}{2\xi}\\ &\times\Bigg\{\frac{t}{\mathbb{Q}^2}\left[3N(\bar u,0)-4(\xi-\rho)(N(\bar u,0))^2+2(\xi-\rho)^2(N(\bar u,0))^3-4\widetilde{N}(\bar u,0)\right]H^{(+)}\\ &-\frac{t}{\mathbb{Q}^2}\partial_\xi\left(\xi\left[2N(\bar u,0)-4\widetilde{N}(\bar u,0)\right]H^{(+)}\right)\\ &+\frac{\bar p_\perp^2}{\mathbb{Q}^2}2\xi^3\partial_\xi^2\left(\xi\left[2N(\bar u,0)-4\widetilde{N}(\bar u,0)\right]H^{(+)}\right)\Bigg\}+O(\text{tw-6})\,, \end{aligned}\tag{4.272}$$

$$\begin{aligned}\mathcal{A}^{00}_{(0),z(\partial-i\Delta)} &= \frac{iQQ'}{\mathbb{Q}^2}\int_{-1}^{1}dx\int_0^1 dv\\ &\times\Bigg\{\frac{t}{\mathbb{Q}^2}\Bigg[\frac{\xi+2\rho}{\xi}N(1,v)+(\xi-\rho)\left(\frac{2(\xi-\rho)}{\xi}-3\right)(N(1,v))^2\\ &\quad+\frac{(\xi-\rho)^2(\xi+\rho)}{\xi}(N(1,v))^3-\frac{2\rho}{\xi}\widetilde{N}(1,v)\Bigg]H^{(+)}\\ &-\frac{t}{\mathbb{Q}^2}\partial_\xi\left(\left[(\xi+\rho)N(1,v)-2\rho\widetilde{N}(1,v)\right]H^{(+)}\right)\\ &+\frac{\bar p_\perp^2}{\mathbb{Q}^2}2\xi^3\partial_\xi^2\left(\left[(\xi+\rho)N(1,v)-2\rho\widetilde{N}(1,v)\right]H^{(+)}\right)\Bigg\}+O(\text{tw-6})\,, \end{aligned}\tag{4.273}$$

$$\begin{aligned}\mathcal{A}^{00}_{(0),\Delta\partial} &= \frac{iQQ'}{\mathbb{Q}^2}\int_{-1}^{1}dx\int_0^1 du\int_0^{\bar u}dv\\ &\times\Bigg\{\frac{-2t}{\mathbb{Q}^2}\left[N(\bar u,v)-(\xi-\rho)(N(\bar u,v))^2-2\widetilde{N}(\bar u,v)\right]H^{(+)}-\frac{t}{\mathbb{Q}^2}\partial_\xi\left(4\xi\widetilde{N}(\bar u,v)H^{(+)}\right)\\ &+\frac{\bar p_\perp^2}{\mathbb{Q}^2}2\xi^3\partial_\xi^2\left(4\xi\widetilde{N}(\bar u,v)H^{(+)}\right)\Bigg\}+O(\text{tw-6})\,, \end{aligned}\tag{4.274}$$

$$\begin{aligned}\mathcal{A}^{00}_{(0),tz\partial} &= \frac{iQQ'}{\mathbb{Q}^2}\int_{-1}^{1}dx\int_0^1 du\int_0^{\bar u}dv\,u\frac{2\xi t}{\mathbb{Q}^2}\left[(N(\bar u,v))^2-2(\xi-\rho)(N(\bar u,v))^3\right]H^{(+)}\\ &+O(\text{tw-6})\,, \end{aligned}\tag{4.275}$$

$$\mathcal{A}^{00}_{(1),z\Delta} = \frac{iQQ'}{Q^2} \int_{-1}^{1} dx \int_0^1 du \int_0^{\bar{u}} dv\, C_{(1),z\Delta}(\bar{u},v)$$
$$\times \left\{ \frac{t}{Q^2} \left[\frac{2\xi}{\xi-\rho+i0} N(\bar{u},v) - 2\xi(\xi-\rho)(N(\bar{u},v))^3 \right] H^{(+)} \right.$$
$$\left. - \frac{t}{Q^2} \frac{2\xi}{\xi-\rho+i0} \partial_\xi \left(\xi N(\bar{u},v) H^{(+)} \right) + \frac{\bar{p}_\perp^2}{Q^2} \frac{2\xi}{\xi-\rho+i0} 2\xi^3 \partial_\xi^2 \left(\xi N(\bar{u},v) H^{(+)} \right) \right\}$$
$$+ O(\text{tw-6})\,, \tag{4.276}$$

$$\mathcal{A}^{00}_{(2),z\Delta} = \frac{iQQ'}{Q^2} \int_{-1}^{1} dx \int_0^1 du \int_0^{\bar{u}} dv\, C_{(2),z\Delta}(\bar{u},v)$$
$$\times \left\{ \frac{t}{Q^2} \left[\frac{2\xi}{\xi-\rho+i0} N(\bar{u},v) + 2\xi(\xi+\rho)(N(\bar{u},v))^3 \right] H^{(+)} \right.$$
$$\left. - \frac{t}{Q^2} \frac{2\xi}{\xi-\rho+i0} \partial_\xi \left(\xi N(\bar{u},v) H^{(+)} \right) + \frac{\bar{p}_\perp^2}{Q^2} \frac{2\xi}{\xi-\rho+i0} 2\xi^3 \partial_\xi^2 \left(\xi N(\bar{u},v) H^{(+)} \right) \right\}$$
$$+ O(\text{tw-6})\,, \tag{4.277}$$

$$\mathcal{A}^{00}_{(1),z\partial} = \frac{iQQ'}{Q^2} \int_{-1}^{1} dx \int_0^1 du \int_0^{\bar{u}} dv\, C_{(1),z\partial}(\bar{u},v)$$
$$\times \left\{ \frac{2t}{Q^2} \left[3N(\bar{u},v) + (\xi-\rho)(N(\bar{u},v))^2 - 2(\xi-\rho)^2(N(\bar{u},v))^3 - 2\widetilde{N}(\bar{u},v) \right] H^{(+)} \right.$$
$$\left. - \frac{t}{Q^2} \partial_\xi \left(4\xi \left[N(\bar{u},v) - \widetilde{N}(\bar{u},v) \right] H^{(+)} \right) + \frac{\bar{p}_\perp^2}{Q^2} 2\xi^3 \partial_\xi^2 \left(4\xi \left[N(\bar{u},v) - \widetilde{N}(\bar{u},v) \right] H^{(+)} \right) \right\}$$
$$+ O(\text{tw-6})\,, \tag{4.278}$$

$$\mathcal{A}^{00}_{(2),z\partial} = \frac{iQQ'}{Q^2} \int_{-1}^{1} dx \int_0^1 du \int_0^{\bar{u}} dv\, C_{(2),z\partial}(\bar{u},v)$$
$$\times \left\{ \frac{2t}{Q^2} \left[3N(\bar{u},v) - (\xi+\rho)(N(\bar{u},v))^2 + 2(\xi-\rho)(\xi+\rho)(N(\bar{u},v))^3 - 2\widetilde{N}(\bar{u},v) \right] H^{(+)} \right.$$
$$\left. - \frac{t}{Q^2} \partial_\xi \left(4\xi \left[N(\bar{u},v) - \widetilde{N}(\bar{u},v) \right] H^{(+)} \right) + \frac{\bar{p}_\perp^2}{Q^2} 2\xi^3 \partial_\xi^2 \left(4\xi \left[N(\bar{u},v) - \widetilde{N}(\bar{u},v) \right] H^{(+)} \right) \right\}$$
$$+ O(\text{tw-6})\,. \tag{4.279}$$

Note that the hard coefficients have been simplified to integrals with respect to u and v of two functions

$$N(\lambda_1,\lambda_2) = \frac{1}{\lambda_1(x-\xi) - \lambda_2(x+\xi) + \xi - \rho + i0}\,,$$
$$\widetilde{N}(\lambda_1,\lambda_2) = \frac{1}{\lambda_1(x-\xi) - \lambda_2(x+\xi)} \ln\left(1 + \frac{\lambda_1(x-\xi) - \lambda_2(x+\xi)}{\xi-\rho+i0} \right), \tag{4.280}$$

weighted with the conformal factors (4.259). The functions N and $\widetilde{N}$ depend not only on u and v, but also on x, ξ and ρ. For brevity, we only make explicit the dependence on u and v. The complexity of the functions (4.259) and (4.280) render long expressions for the hard coefficients of $\mathcal{A}^{00}$ that are difficult to manage. Consequently, it seems more practical to express the subamplitudes of $\mathcal{A}^{00}$ by means of hard coefficients given through integrals over u and v with the conformal weights (4.259), as illustrated in Eqs. (4.272) to (4.279). In fact, the phenomenological results presented in Sect. 4.10 combine both analytical and numerical calculations of the different components of $\mathcal{A}^{00}$.

The subamplitudes $\mathcal{A}^{00}_{(1),z\Delta}$ and $\mathcal{A}^{00}_{(2),z\Delta}$, vid. Eqs. (4.276, 4.277), contain an apparent divergence in the DVCS limit ($\rho \to \xi$) given by the factor:

$$\frac{iQQ'}{\mathbb{Q}^2}\frac{2\xi}{\xi-\rho+i0} = \frac{iQ}{\mathbb{Q}}\sqrt{\frac{2\xi}{\xi-\rho+i0}}\,. \tag{4.281}$$

The origin of this term is the Fourier transform

$$i\int d^4z\, e^{iq'z}\frac{z^\mu\partial^\nu}{-z^2+i0}(i\Delta\partial)\left[e^{-i\ell z}\right]_{\text{LT}} = -8\pi^2 i\left[\frac{q^\nu q'^\mu-\Delta^\nu\ell^\mu}{(a+b+c)^2}(\Delta\ell)+aq^\nu q'^\mu\mathbb{Q}^2 I_{1,3}\right] + O(\text{tw-6})\,, \tag{4.282}$$

where $I_{1,3}$ has been defined as

$$I_{1,3} = \int_0^1 dw\,\frac{w}{(aw^2+bw+c)^3}\,,\quad a=\ell^2\,,\quad b=-2q'\ell\,,\quad c=Q'^2+i0\,. \tag{4.283}$$

This integral contributes to twist-4 with a term proportional to $1/c = 1/(Q'^2+i0)$, later producing the dependence on $1/(\xi-\rho+i0)$.

Fortunately, in this limit the integration of such term with respect to u and $v=\bar{u}w$ vanishes thanks to:

$$\int_0^1 dw\,\frac{1}{x-\xi-w(x+\xi)+i0}\int_0^1 du\,\left[C_{(1),z\Delta}(\bar{u},\bar{u}w)+C_{(2),z\Delta}(\bar{u},\bar{u}w)\right]=0\,. \tag{4.284}$$

With this remark, we conclude the calculation of $\mathcal{A}^{00}$ which completes the set of helicity amplitudes required for describing DDVCS off a spin-0 target.

4.9 Collection of final results for DDVCS

In this section, we gather the final expressions for the helicity amplitudes that parameterize the Compton tensor for a general two-photon scattering off a spin-0 target, cf. Eq. (4.89). Remember that these amplitudes are related to the helicity-dependent CFFs by

$$\mathcal{A}^{AB} = \frac{1}{2}\mathcal{H}^{AB}\,, \tag{4.285}$$

with the condition $\mathcal{H}^{++}|_{\mathrm{LT}} = \mathcal{H}|_{\mathrm{Eq.\,(1.93)}}$. The amplitudes/CFFs are:

$$
\begin{aligned}
\mathcal{A}^{++} = \frac{1}{2}\int_{-1}^{1} dx \Bigg\{ &-\left(1-\frac{t}{2Q^2}+\frac{t(\xi-\rho)}{Q^2}\partial_\xi\right)\frac{H^{(+)}}{x-\rho+i0} \\
&+\frac{t}{\xi Q^2}\Bigg[\mathbb{P}_{(\mathrm{i})}+\mathbb{P}_{(\mathrm{ii})}-\frac{\widetilde{\mathbb{P}}_{(\mathrm{i})}-\widetilde{\mathbb{P}}_{(\mathrm{iii})}}{2}-\frac{\xi(J+L)}{2} \\
&\qquad -\frac{\xi}{x+\xi}\left(\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right)-\frac{\xi+\rho}{2\xi}\ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right)-\widetilde{\mathbb{P}}_{(\mathrm{i})}\right)\Bigg]H^{(+)} \\
&-\frac{t}{Q^2}\partial_\xi\Bigg[\Bigg(\mathbb{P}_{(\mathrm{i})}+\mathbb{P}_{(\mathrm{ii})}-\frac{\xi L}{2}-\frac{\xi}{x+\xi}\Bigg(\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right) \\
&\qquad -\frac{\xi+\rho}{2\xi}\ln\left(\frac{-\xi-\rho+i0}{\xi-\rho+i0}\right)-\widetilde{\mathbb{P}}_{(\mathrm{i})}\Bigg)\Bigg)H^{(+)}\Bigg] \\
&+\frac{\bar{p}_\perp^2}{Q^2}2\xi^3\partial_\xi^2\left[\left(\mathbb{P}_{(\mathrm{i})}+\mathbb{P}_{(\mathrm{ii})}-\frac{\widetilde{\mathbb{P}}_{(\mathrm{i})}-\widetilde{\mathbb{P}}_{(\mathrm{iii})}}{2}-\frac{\xi L}{2}+\ln\left(\frac{x-\rho+i0}{x-\xi+i0}\right)\right)H^{(+)}\right]\Bigg\} \\
&+O(\text{tw-6})\,,
\end{aligned}
\tag{4.286}
$$

$$
\mathcal{A}^{+-} = \frac{\bar{p}_\perp^2}{Q^2}4(\xi^2\partial_\xi)^2\int_{-1}^{1}\frac{dx}{2\xi}\left(\widetilde{\mathbb{P}}_{(\mathrm{iii})}-\widetilde{\mathbb{P}}_{(\mathrm{i})}+2\ln\left(\frac{x-\rho+i0}{-2\xi}\right)\right)H^{(+)}+O(\text{tw-6})\,,
\tag{4.287}
$$

$$
\mathcal{A}^{+0} = \frac{-iQ'|\bar{p}_\perp|}{\sqrt{2}Q^2}\int_{-1}^{1}dx\,2\xi^2\partial_\xi\left(\frac{1}{x-\xi}\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right)H^{(+)}\right)+O(\text{tw-5})\,,
\tag{4.288}
$$

$$
\begin{aligned}
\mathcal{A}^{0+} = \frac{-Q|\bar{p}_\perp|}{\sqrt{2}Q^2}\int_{-1}^{1}dx\,2\xi^2\partial_\xi\Bigg(&\frac{H^{(+)}}{2\xi}\Bigg[\frac{\xi+\rho}{x+\xi}\ln\left(\frac{x-\rho+i0}{-\xi-\rho+i0}\right) \\
&+\frac{\xi-\rho}{x+\xi}\ln\left(\frac{x(1-2t/Q^2)-\rho+i0}{-\xi(1-2t/Q^2)-\rho+i0}\right)\Bigg]\Bigg)+O(\text{tw-5})\,,
\end{aligned}
\tag{4.289}
$$

and

$$
\mathcal{A}^{00} = \mathcal{A}^{00}_{(0),z\partial}+\mathcal{A}^{00}_{(0),z(\partial-i\Delta)}+\mathcal{A}^{00}_{(0),\Delta\partial}+\mathcal{A}^{00}_{(0),tz\partial}+\mathcal{A}^{00}_{(1),z\Delta}+\mathcal{A}^{00}_{(2),z\Delta}+\mathcal{A}^{00}_{(1),z\partial}+\mathcal{A}^{00}_{(2),z\partial}\,.
\tag{4.290}
$$

These results have been given by means of integrals J and L presented in Eqs. (4.199), (4.206) and (4.197), as well as the functions

$$
\begin{aligned}
\mathbb{P}_{(\mathrm{i})}(x/\xi,\rho/\xi) &= \frac{\xi-\rho}{x-\xi}\mathrm{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right), \\
\widetilde{\mathbb{P}}_{(\mathrm{i})}(x/\xi,\rho/\xi) &= -\frac{\xi-\rho}{x-\xi}\ln\left(\frac{x-\rho+i0}{\xi-\rho+i0}\right), \\
\mathbb{P}_{(\mathrm{ii})}(x/\xi,\rho/\xi) &= \frac{\xi-\rho}{x+\xi}\left[\mathrm{Li}_2\left(-\frac{x-\xi}{\xi-\rho+i0}\right)-(x\to-\xi)\right],
\end{aligned}
$$

$$\widetilde{\mathbb{P}}_{(\text{iii})}(x/\xi,\rho/\xi) = -\frac{\xi+\rho}{x+\xi}\ln\left(\frac{x-\rho+i0}{-\xi-\rho+i0}\right). \tag{4.291}$$

Also, the components of $\mathcal{A}^{00}$ have been detailed in Eqs. (4.272) to (4.279). The DVCS and TCS limits of the above formulation have been discussed in the previous sections, as well.

4.10 Numerical estimates of the kinematic twist corrections

In this section we present the numerical calculation of the helicity amplitudes that describe a pseudo-scalar target in the general two-photon scattering (i.e. DDVCS), focusing on the pion target. We emphasize that a complete phenomenological study of higher-twist corrections can only be done through cross-sections and other observables. Such a study is, however, beyond the scope of this doctoral thesis and is left for future phenomenology-oriented projects.

In order to obtain numerical estimates, we numerically convolute the calculated coefficient functions with a pion GPD model. For the later, we employ the phenomenological model of Ref. [124] based on the double distribution representation [125]. A crude simplification of this model is used for the $\mathcal{A}^{00}$ amplitude:

$$H(x,\xi) = A\,x(1-x^2)(1+B\xi^2) \tag{4.292}$$

where the coefficients A and B are fitted to the model of [124] for each value of t we are using in the numerical estimate. Namely, we have $A = 1.8$, $B = -0.2$ for $t = -0.1\,\text{GeV}^2$ and $A = 1.1$, $B = -0.2$ for $t = -0.6\,\text{GeV}^2$. Such a special treatment is required due to severe numerical instabilities spoiling the implementation of $\mathcal{A}^{00}$. The source of these instabilities comes from the elements proportional to $1/(u(x-\xi) - uw(x+\xi) + \xi - \rho + i0)^3$, which we are unable to completely calculate analytically, and the numerical integration struggles due to the power 3 in the denominator.

The amplitudes are depicted as functions of the following ratio of virtualities:

$$\mathfrak{F} = \frac{Q^2 - Q'^2}{Q^2 + Q'^2}. \tag{4.293}$$

This ratio is defined in such a way that the DVCS and TCS limits correspond to the extreme right ($\mathfrak{F} = +1$) and extreme left ($\mathfrak{F} = -1$) points, respectively. We keep the values of ξ, $\mathbb{Q}^2$ and t fixed, meaning that a given $\mathfrak{F}$ corresponds to a specific value of Björken variable, cf. Eq. (4.29). Otherwise, the different points in Figs. 4.1 to 4.5 would correspond to different kinematics at which GPD is probed, making the interpretation of results cumbersome.

As proven in previous sections, the helicity amplitude $\mathcal{A}^{++}$ contains a leading-twist-like component, namely

$$-\frac{1}{2}\int_{-1}^{1} dx\,\frac{1}{x-\rho+i0}H^{(+)}, \tag{4.294}$$

followed by a series of terms that are weighted with dumping factors $|t|/\mathcal{Q}^2$ and $\xi^2|\bar{p}_\perp|^2/\mathcal{Q}^2$. Taking into account that for fixed ξ we may write

$$\rho = \xi\,\mathfrak{F} + O(|t|/\mathcal{Q}^2)\,, \tag{4.295}$$

then we can effectively drop the t-dependence of ρ for these higher-twist contributions. In other words, the t-dependence of ρ can be translated to terms whose weight in the calculation is of the same order as the next twist and, therefore, can be disregarded. However, for the term (4.294) ρ must be kept in exact form (4.141).

The Feynman-$i0$ present in the hard-coefficient functions of the helicity amplitudes is dealt with numerically, i.e. we take $|i0| \ll \xi$ and confirm the stability of the computation as the $|i0| \to 0$ limit is approached. The value of $i0$ for $\mathcal{A}^{++}$, $\mathcal{A}^{+-}$, $\mathcal{A}^{+0}$ and $\mathcal{A}^{0+}$ is $i10^{-4}$. We checked by changing this value that its finiteness introduces a bias of the order 5% with respect to the absolute magnitude of the amplitude. For $\mathcal{A}^{00}$ we have $i0.05$, introducing a $\sim$ 20% bias. Such a large value is forced by the unstable calculation we experience when dealing with this amplitude.

Figure 4.1 illustrates the behaviour of $\mathcal{A}^{++}$ as a function of $\mathfrak{F}$ for the GPD model of [124], the skewness variable $\xi = 0.2$, the energy scale $\mathcal{Q}^2 = 1.9$ GeV^2 (corresponding to the reference scale of the used GPD model) and two values of t: -0.1 GeV^2 and -0.6 GeV^2. Results for both leading twist (i.e. twist-2) and higher-twist (up to twist-4) calculations are visible. As expected, for high $|t|$ the later shows sizable corrections to LT calculation that range between 25% and 50%, also suggesting a sizable effect on the observables related to DDVCS, DVCS and TCS. We note that higher-twist corrections break the simple relation between DVCS and TCS amplitudes one observes at LO and LT [126]: $\mathcal{A}^{++}_{\text{DVCS}} \overset{\text{LO,LT}}{=} \left(\mathcal{A}^{++}_{\text{TCS}}\right)^*$. This observation is crucial for future attempts of proving GPD universality through measurements of DVCS and TCS, see for instance Ref. [87].

The amplitudes $\mathcal{A}^{+-}$, $\mathcal{A}^{+0}$ and $\mathcal{A}^{0+}$ are depicted in Figs. 4.2, 4.3, 4.4, respectively, for the same values of ξ, $\mathcal{Q}^2$ and t as $\mathcal{A}^{++}$. These amplitudes are null at LT. Our numerical estimate agrees with the expectation that these amplitudes stay suppressed with respect to $\mathcal{A}^{++}$ also in the higher-twist calculation.

Finally, $\mathcal{A}^{00}$ is shown in Fig. 4.5. We remind that in this case a crude simplification of the model based on Ref. [124] was used. Also, the result should be treated as preliminary because of numerical instabilities imposing the use of a high value of $i0$. A more sophisticated numerical approach will be taken in a future work. We note that the evaluation of this amplitude will pose a challenge for DDVCS phenomenology, but it will not affect DVCS and TCS studies as $\mathcal{A}^{00}$ does not contribute to the cross-sections of these two processes.

4.11 Summary and conclusions

In this chapter we have delivered a complete formalism to study DVCS, TCS and DDVCS off scalar and pseudo-scalar targets at LO including kinematic twist-3 and twist-4 corrections. We started by providing a parameterization based on helicity amplitudes (which can be connected to CFFs) for the Compton tensor of a general two-photon scattering, regardless of the spin of the hadron, see Eq. (4.72). This expression was particularized for the spin-0 target and reduced to five tensor structures and helicity amplitudes, Eq. (4.89), thanks to the constrains imposed by parity

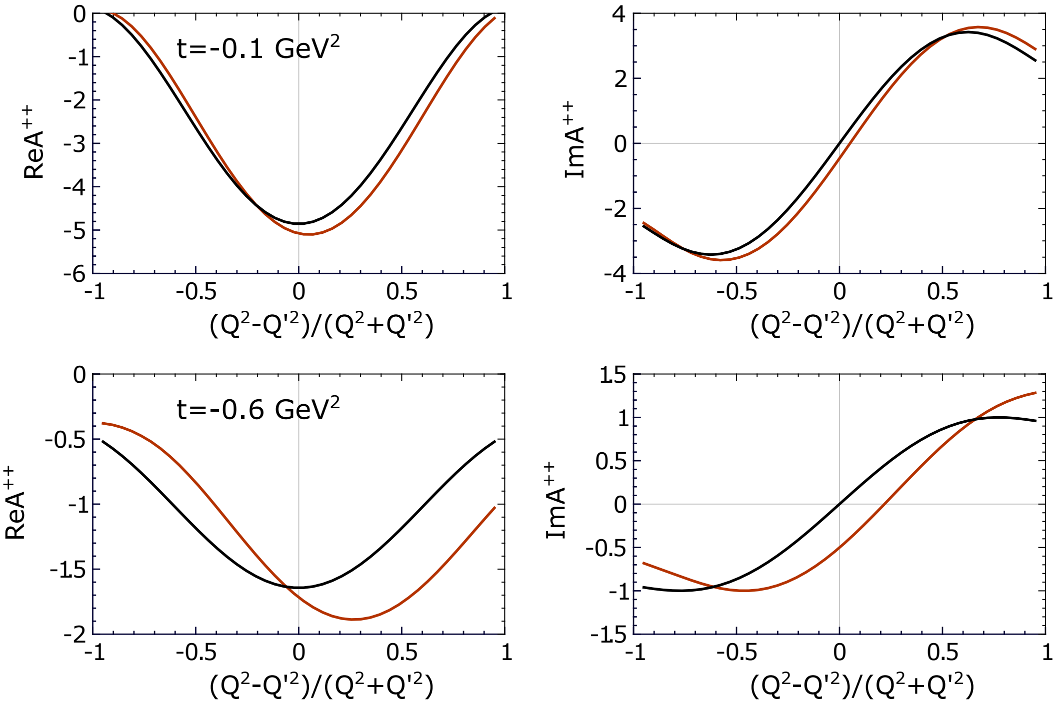

Fig. 4.1. Real (left) and imaginary (right) parts of the transverse-helicity conserving amplitude, $\mathcal{A}^{++}$, for $\xi = 0.2$, $Q^2 = 1.9$ GeV^2 and two values of t: $-0.1\,\text{GeV}^2$ (first row) and $-0.6\,\text{GeV}^2$ (second row); as a function of virtualities ratio. Black and red lines represent the LT and higher-twist results, respectively. The results have been obtained for the GPD model of Ref. [124].

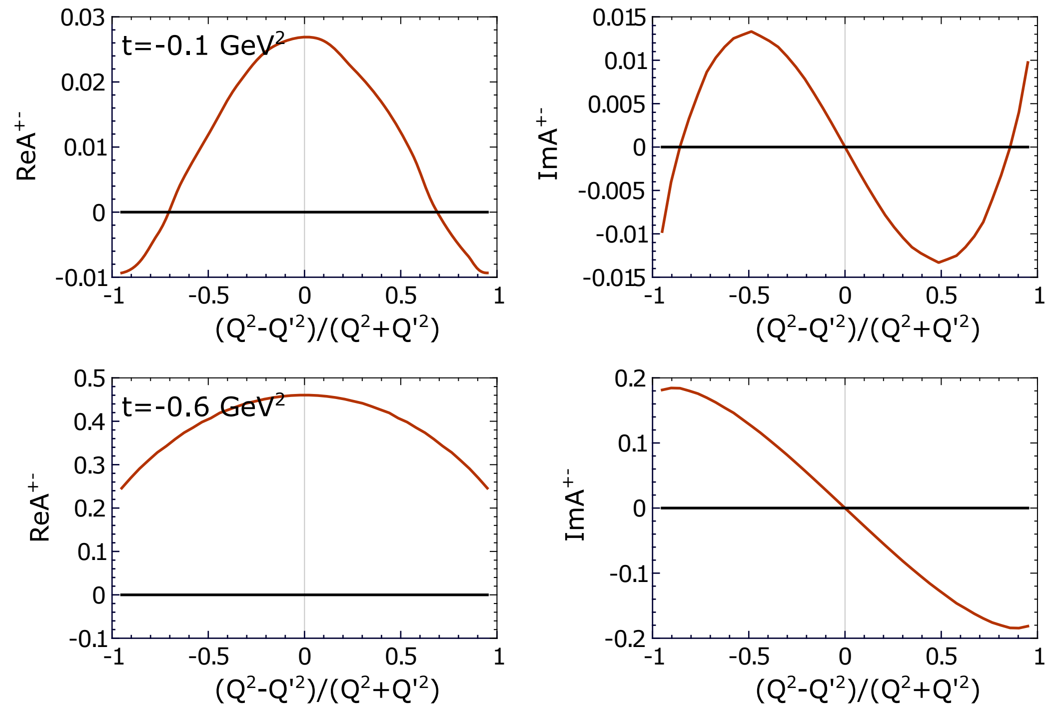

Fig. 4.2. Numerical estimate for the transverse-helicity flip amplitude, $\mathcal{A}^{+-}$. For a further description see the caption of Fig. 4.1.

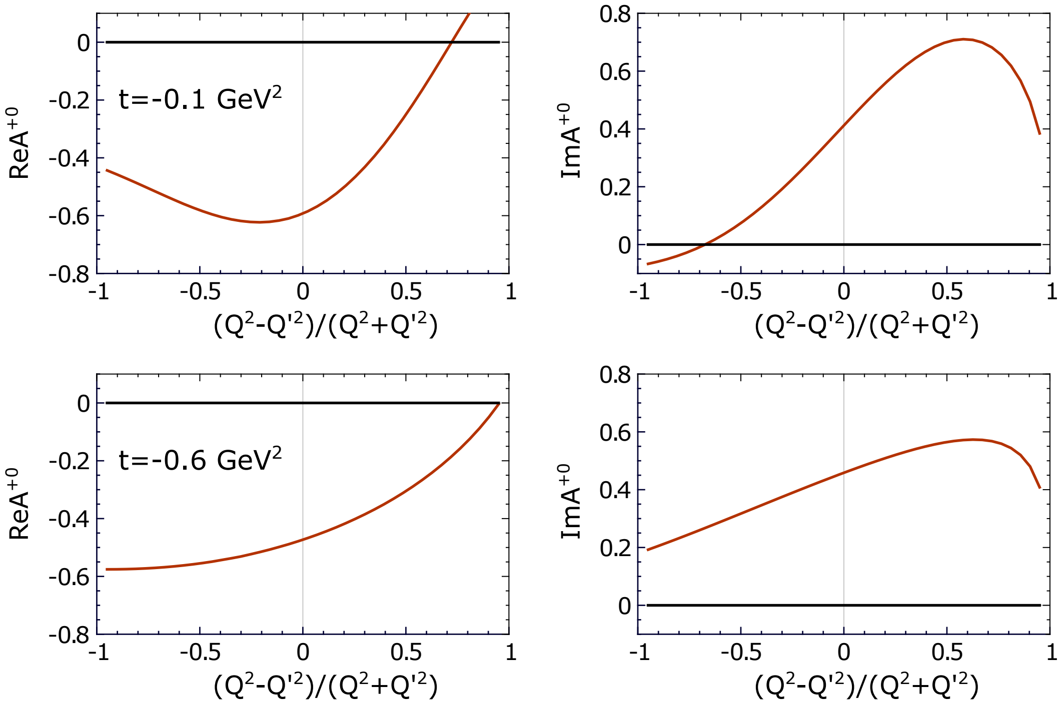

Fig. 4.3. Numerical estimate for the transverse-to-longitudinal helicity flip amplitude, $\mathcal{A}^{+0}$. For a further description see the caption of Fig. 4.1.

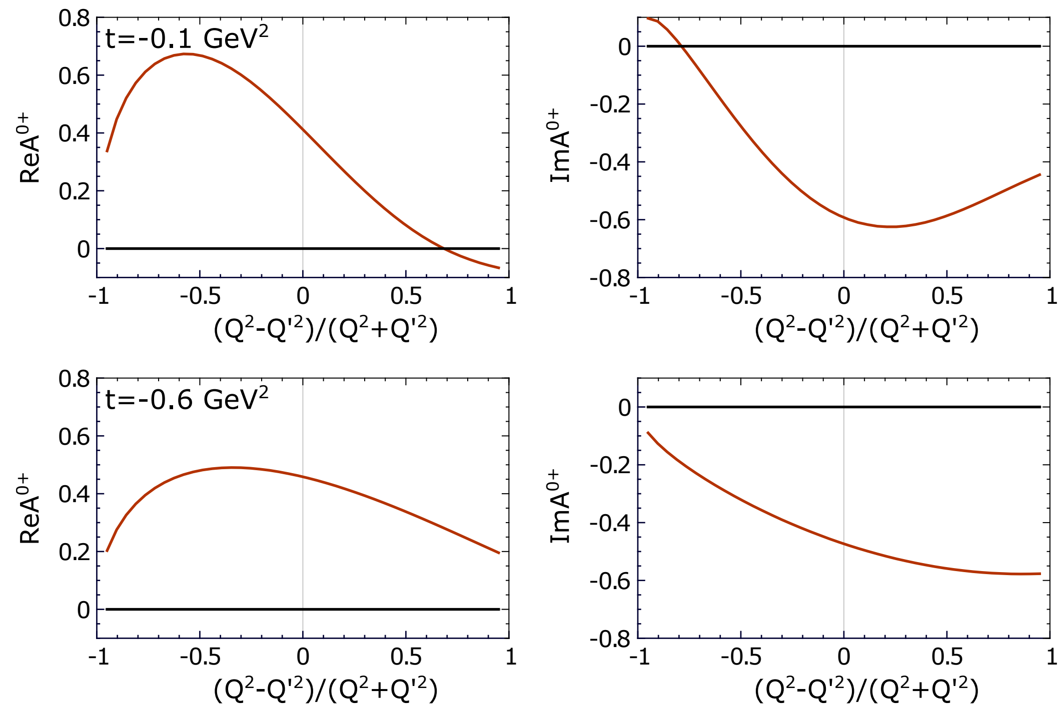

Fig. 4.4. Numerical estimate for the longitudinal-to-transverse helicity flip amplitude, $\mathcal{A}^{0+}$. For a further description see the caption of Fig. 4.1.

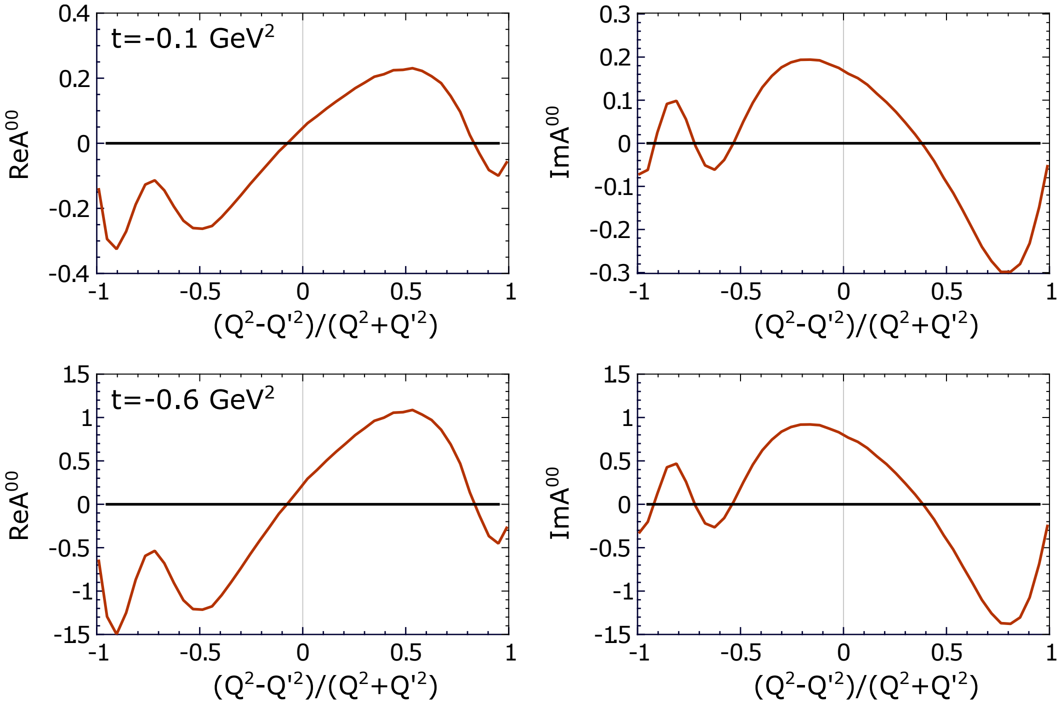

Fig. 4.5. Numerical estimate for the longitudinal-helicity conserving amplitude, $\mathcal{A}^{00}$, obtained for a toy model given by Eq. (4.292). For the description of the used color code and kinematics see the caption of Fig. 4.1.

conservation. From there, we read out a series of projectors onto the helicity amplitudes which were applied to the conformal operator-product expansion of the Compton tensor developed by Braun, Ji and Mansahov, and presented in Eq. (3.163) for its vector component.

With this set-up we were able to compute the twist expansion of the different helicity amplitudes and provide numerical estimates for these amplitudes/CFFs. Their magnitude advocates for the correctness of the twist expansion and suggests that these corrections do not endanger collinear factorization. Moreover, these estimates suggest that the kinematic corrections are sizable enough to affect observables and, therefore, be measurable in both current and future experimental facilities. We stress out that a complete phenomenological impact study should include the calculation of the cross-section and other observables such as beam- and target-spin asymmetries. Such a study is beyond the goals of this doctoral project.

5

Overview, conclusions and future prospects

In this final chapter, we present a comprehensive summary of the works taken over in this doctoral project that were detailed in the previous chapters. We will highlight the main conclusions and how the scientific community may benefit from this research work. Examples of the future studies that can be developed using our results are also considered.

In Ch. 1 we outline the concepts of QCD and of collinear factorization through several processes, both inclusive (DIS) and exclusive (DVCS, TCS and DDVCS). The collinear factorization is a consequence of the light-cone dominance that characterizes the aforementioned reactions in the Björken limit. Factorization for processes where the longitudinal momentum is dominant happens via parton distribution functions (PDFs) in the cross-section of inclusive processes, and by generalized parton distributions (GPDs) in the amplitude of exclusive scatterings. PDFs depend on one variable only (the Björken variable x_B) which coincides with the average fraction x of the hadron longitudinal momentum that is carried by the active parton in the interaction. Conversely, GPDs are three-dimensional distributions depending on: the already introduced x, the skewness ξ that provides information on the change of the hadron longitudinal momentum, and the Mandelstam variable t. We dedicate our efforts to the latter as GPDs possess unique properties. They are connected to the energy-momentum tensor giving access to the "mechanical" properties of hadrons and the total angular momentum of partons via Ji's sum rule, allowing us to tackle the so-called *hadron spin puzzle*. Also, by Fourier transform with respect to the transverse momentum transfer to the target, GPDs enable a 3D picture combining spatial and momentum features, which receives the name of hadron tomography.

At the lowest order approximation (leading order (LO) in α_s and leading twist (LT)), the access to GPDs through DVCS and TCS reactions is restricted to $x = \pm\xi$. For the purpose of studying the region $x \neq \xi$, in Ch. 2 we consider DDVCS off a nucleon target, through the electroproduction of a muon pair. We compute the amplitudes that contribute to the cross-section of this electroproduction and implement them in the `PARTONS` (for numerical estimates of observables) and `EpIC` (for Monte Carlo simulations) softwares. The cross-section, their cosine moments and the single beam-spin asymmetry are the selected observables for JLab and EIC kinematics. Their magnitude suggests the possibility of measurements of DDVCS at current and future experimental facilities. We also demonstrate that DDVCS serves as a laboratory for addressing GPD model dependence, tackling the so-called *deconvolution problem*.

The presented results were obtained assuming that the kinematics of JLab and the future EIC is a good realization of the infinite virtuality limit, so that kinematic corrections proportional to powers of $|t|/\mathbb{Q}^2$ and $M^2/\mathbb{Q}^2$ (with M the hadron mass and $\mathbb{Q}^2$ the energy scale of the process) are negligible. In practice, satisfying this limit requires to introduce some rejection cuts in the experimental data, so that $|t|/\mathbb{Q}^2 \ll 1$ and $M^2/\mathbb{Q}^2 \ll 1$. To increase the set of useful data for GPD extraction and tomography, as well as to provide a more accurate theoretical framework, we consider including such corrections. For that purpose, it is required to relax the condition of an infinite energy scale. In such a case, the description of the aforementioned processes needs to include quark and gluon operators which are not evaluated on the light-cone. In this regard, Ch. 3 is dedicated to an introduction to conformal field theory (CFT) and its methods, with special emphasis in the so-called shadow-operator formalism. CFT is of interest due to the predictability of Green functions, which together with the aforementioned formalism, settles the grounds for the operator-product expansion developed by Braun, Ji and Mansahov for the product of two currents. This is of a special interest in QCD, as the main object carrying the information on the hadron structure is the Compton tensor, which is defined by matrix elements of the time-ordering of two spin-1 conserved quark currents. This operator-product expansion around the light-cone prompts the kinematic power corrections mentioned earlier. They also receive the name of kinematic twists. Their relevance is crucial to enlarge the useful data sets for GPD extraction and specially for hadron tomography as it requires a Fourier transform on the transverse momentum transfer $\Delta_\perp$ (related to $t = 2(nn')\Delta^+\Delta^- + \Delta_\perp^2$ with the light-cone coordinates notation used throughout this manuscript).

Regarding the kinematic higher-twist corrections, in Ch. 4 we took over the calculation following the conformal techniques detailed in Ch. 3 for the case of DDVCS off a spin-0 target. The advantage of a spinless hadron is that the number of GPDs and of Compton form factors (CFFs) is minimal: one GPD, one CFF at LT and five when twist corrections are included. Therefore, from the point of view of extraction from data the number of measurements required to constrain the GPD and CFFs is less than for other hadrons of higher spin. In Ch. 4 we give a general parameterization of the Compton tensor for two-photon scatterings off a target of arbitrary spin and particularize it for scalar and pseudo-scalar targets. The parameterization is given by means of helicity amplitudes that can be related to (helicity-dependent) CFFs. From the conformal operator-product expansion introduced in Ch. 3 we are able to identify these CFFs and calculate them up to twist-4 accuracy, including DVCS and TCS expressions obtained as limiting cases of the DDVCS formulation. Numerical estimates of these CFFs are provided for a pion target and suggest that the collinear factorization is still applicable when twist corrections are taken into account. The helicity amplitudes described in Ch. 4 are the building blocks of a future analysis on the phenomenology of DVCS, TCS and DDVCS considering observables, for which the computation of the cross-section is required.

The works detailed in this thesis provide the theoretical support for experimental proposals at JLab and EIC, among other facilities, looking for measuring DDVCS and related processes. The studies presented here can be used to improve the GPD extraction by including data in a larger range of t from measurements of DVCS and from a future TCS off a spin-0 target. On top of this, higher-twist corrections also allow for the extension of the range in Q^2 which can be useful for instance in the analysis of the "mechanical" properties of partonic systems (pressure, shear forces, etc) [48].

Bibliography

[1] W. Heisenberg. Über den Bau der Atomkerne. I. *Z. Physik* 77 (1932), pp. 1–11. DOI: 10.1007/BF01342433.

[2] M. Gell-Mann. A Schematic Model of Baryons and Mesons. *Phys. Lett.* 8 (1964), pp. 214–215. DOI: 10.1016/S0031-9163(64)92001-3.

[3] G. Zweig. An SU(3) model for strong interaction symmetry and its breaking. Version 2. *Developments in the quark theory of hadrons, volume 1: 1964 - 1978*. Ed. by D. B. Lichtenberg and S. P. Rosen. Feb. 1964, pp. 22–101. DOI: 10.17181/CERN-TH-412.

[4] R. P. Feynman and G. Speisman. Proton-Neutron Mass Difference. *Phys. Rev.* 94 (2 Apr. 1954), pp. 500–500. DOI: 10.1103/PhysRev.94.500.

[5] V. E. Barnes et al. Observation of a Hyperon with Strangeness Minus Three. *Phys. Rev. Lett.* 12 (8 Feb. 1964), pp. 204–206. DOI: 10.1103/PhysRevLett.12.204.

[6] O. W. Greenberg. Spin and Unitary-Spin Independence in a Paraquark Model of Baryons and Mesons. *Phys. Rev. Lett.* 13 (20 Nov. 1964), pp. 598–602. DOI: 10.1103/PhysRevLett.13.598.

[7] G. Hanson et al. Evidence for Jet Structure in Hadron Production by e^+e^- Annihilation. *Phys. Rev. Lett.* 35 (24 Dec. 1975), pp. 1609–1612. DOI: 10.1103/PhysRevLett.35.1609.

[8] G. Hanson et al. Hadron production by e^+e^- annihilation at center-of-mass energies between 2.6 and 7.8 GeV. II. Jet structure and related inclusive distributions. *Phys. Rev. D* 26 (5 Sept. 1982), pp. 991–1012. DOI: 10.1103/PhysRevD.26.991.

[9] D. P. Barber et al. Discovery of Three-Jet Events and a Test of Quantum Chromodynamics at PETRA. *Phys. Rev. Lett.* 43 (12 Sept. 1979), pp. 830–833. DOI: 10.1103/PhysRevLett.43.830.

[10] C. Berger et al. A study of multi-jet events in e+e- annihilation. *Physics Letters B* 97.3 (1980), pp. 459–464. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(80)90640-1.

[11] R. Brandelik et al. Evidence for planar events in e+e- annihilation at high energies. *Physics Letters B* 86.2 (1979), pp. 243–249. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(79)90830-X.

[12] W. Bartel et al. Observation of planar three-jet events in e+e- annihilation and evidence for gluon bremsstrahlung. *Physics Letters B* 91.1 (1980), pp. 142–147. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(80)90680-2.

[13] L. Faddeev and V. Popov. Feynman diagrams for the Yang-Mills field. *Physics Letters B* 25.1 (1967), pp. 29–30. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(67)90067-6.

[14] D. J. Gross and F. Wilczek. Ultraviolet Behavior of Non-Abelian Gauge Theories. *Phys. Rev. Lett.* 30 (26 June 1973), pp. 1343–1346. DOI: 10.1103/PhysRevLett.30.1343.

[15] H. D. Politzer. Reliable Perturbative Results for Strong Interactions? *Phys. Rev. Lett.* 30 (26 June 1973), pp. 1346–1349. DOI: 10.1103/PhysRevLett.30.1346.

[16] R. P. Feynman. Very High-Energy Collisions of Hadrons. *Phys. Rev. Lett.* 23 (24 Dec. 1969), pp. 1415–1417. DOI: 10.1103/PhysRevLett.23.1415.

[17] M. Breidenbach et al. Observed Behavior of Highly Inelastic Electron-Proton Scattering. *Phys. Rev. Lett.* 23 (16 Oct. 1969), pp. 935–939. DOI: 10.1103/PhysRevLett.23.935.

[18] R. Hofstadter and R. W. McAllister. Electron Scattering from the Proton. *Phys. Rev.* 98 (1 Apr. 1955), pp. 217–218. DOI: 10.1103/PhysRev.98.217.

[19] Y. L. Dokshitzer. Calculation of the Structure Functions for Deep Inelastic Scattering and e+ e- Annihilation by Perturbation Theory in Quantum Chromodynamics. *Sov. Phys. JETP* 46 (1977), pp. 641–653.

[20] V. N. Gribov and L. N. Lipatov. Deep inelastic e p scattering in perturbation theory. *Sov. J. Nucl. Phys.* 15 (1972), pp. 438–450.

[21] G. Altarelli and G. Parisi. Asymptotic freedom in parton language. *Nuclear Physics B* 126.2 (1977), pp. 298–318. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(77)90384-4.

[22] R. Boussarie et al. TMD Handbook (Apr. 2023). arXiv: 2304.03302 [hep-ph].

[23] D. Müller et al. Wave functions, evolution equations and evolution kernels from light ray operators of QCD. *Fortsch. Phys.* 42 (1994), pp. 101–141. DOI: 10.1002/prop.2190420202.

[24] X.-D. Ji. Deeply virtual Compton scattering. *Phys. Rev. D* 55 (1997), pp. 7114–7125. DOI: 10.1103/PhysRevD.55.7114.

[25] A. V. Radyushkin. Symmetries and structure of skewed and double distributions. *Phys. Lett. B* 449 (1999), pp. 81–88. DOI: 10.1016/S0370-2693(98)01584-6.

[26] M. Diehl. Generalized parton distributions. *Phys. Rep.* 388.2-4 (Dec. 2003), pp. 41–277. DOI: 10.1016/j.physrep.2003.08.002.

[27] J. Ashman et al. An Investigation of the Spin Structure of the Proton in Deep Inelastic Scattering of Polarized Muons on Polarized Protons. *Nucl. Phys. B* 328 (1989). Ed. by V. W. Hughes and C. Cavata, p. 1. DOI: 10.1016/0550-3213(89)90089-8.

[28] M. Burkardt. Impact parameter space interpretation for generalized parton distributions. *Int. J. Mod. Phys. A* 18 (2003), pp. 173–208. DOI: 10.1142/S0217751X03012370.

[29] R. Abdul Khalek et al. Science Requirements and Detector Concepts for the Electron-Ion Collider: EIC Yellow Report. *Nucl. Phys. A* 1026 (2022), p. 122447. DOI: 10.1016/j.nuclphysa.2022.122447.

[30] D. P. Anderle et al. Electron-ion collider in China. *Front. Phys. (Beijing)* 16.6 (2021), p. 64701. DOI: 10.1007/s11467-021-1062-0.

[31] J. L. A. Fernandez et al. A Large Hadron Electron Collider at CERN Report on the Physics and Design Concepts for Machine and Detector. *Journal of Physics G: Nuclear and Particle Physics* 39.7 (June 2012), p. 075001. DOI: 10.1088/0954-3899/39/7/075001.

[32] J. Arrington et al. The solenoidal large intensity device (SoLID) for JLab 12 GeV. *J. Phys. G* 50.11 (2023), p. 110501. DOI: 10.1088/1361-6471/acda21.

[33] A. Pedrak et al. Hard photoproduction of a diphoton with a large invariant mass. *Phys. Rev. D* 96.7 (2017). [Erratum: Phys.Rev.D 100, 039901 (2019)], p. 074008. DOI: 10.1103/PhysRevD.96.074008.

[34] A. Pedrak et al. Electroproduction of a large invariant mass photon pair. *Phys. Rev. D* 101.11 (2020), p. 114027. DOI: 10.1103/PhysRevD.101.114027.

[35] O. Grocholski et al. Collinear factorization of diphoton photoproduction at next to leading order. *Phys. Rev. D* 104.11 (2021), p. 114006. DOI: 10.1103/PhysRevD.104.114006.

[36] O. Grocholski et al. Phenomenology of diphoton photoproduction at next-to-leading order. *Phys. Rev. D* 105.9 (2022), p. 094025. DOI: 10.1103/PhysRevD.105.094025.

[37] D. Y. Ivanov et al. Probing chiral odd GPD's in diffractive electroproduction of two vector mesons. *Phys. Lett. B* 550 (2002), pp. 65–76. DOI: 10.1016/S0370-2693(02)02856-3.

[38] R. Boussarie et al. Exclusive photoproduction of a $\gamma\,\rho$ pair with a large invariant mass. *JHEP* 02 (2017). [Erratum: JHEP 10, 029 (2018)], p. 054. DOI: 10.1007/JHEP02(2017)054.

[39] G. Duplančić et al. (Feb. 2023). Probing chiral-even and chiral-odd leading twist quark generalised parton distributions through the exclusive photoproduction of a $\gamma\rho$ pair. arXiv: hep-ph/2302.12026 [hep-ph].

[40] V. Bertone et al. Deconvolution problem of deeply virtual Compton scattering. *Phys. Rev. D* 103.11 (2021), p. 114019. DOI: 10.1103/PhysRevD.103.114019.

[41] E. R. Berger, M. Diehl, and B. Pire. Time - like Compton scattering: Exclusive photoproduction of lepton pairs. *Eur. Phys. J.* C23 (2002), pp. 675–689. DOI: 10.1007/s100520200917.

[42] A. V. Belitsky and D. Müller. Probing generalized parton distributions with electroproduction of lepton pairs off the nucleon. *Phys. Rev. D* 68 (2003), p. 116005. DOI: 10.1103/PhysRevD.68.116005.

[43] M. Guidal and M. Vanderhaeghen. Double Deeply Virtual Compton Scattering off the Nucleon. *Phys. Rev. Lett.* 90 (1 Jan. 2003), p. 012001. DOI: 10.1103/PhysRevLett.90.012001.

[44] H. Moutarde, P. Sznajder, and J. Wagner. Unbiased determination of DVCS Compton Form Factors. *Eur. Phys. J. C* 79.7 (2019), p. 614. DOI: 10.1140/epjc/s10052-019-7117-5.

[45] K. G. Wilson. Non-Lagrangian Models of Current Algebra. *Phys. Rev.* 179 (5 Mar. 1969), pp. 1499–1512. DOI: 10.1103/PhysRev.179.1499.

[46] V. M. Braun, Y. Ji, and A. N. Manashov. Two-photon processes in conformal QCD: resummation of the descendants of leading-twist operators. *JHEP* 03 (2021), p. 051. DOI: 10.1007/JHEP03(2021)051.

[47] V. M. Braun, Y. Ji, and A. N. Manashov. Next-to-leading-power kinematic corrections to DVCS: a scalar target. *JHEP* 01 (2023), p. 078. DOI: 10.1007/JHEP01(2023)078.

[48] H. Dutrieux et al. Phenomenological assessment of proton mechanical properties from deeply virtual Compton scattering. *Eur. Phys. J. C* 81.4 (2021), p. 300. DOI: 10.1140/epjc/s10052-021-09069-w.

[49] B. Berthou et al. PARTONS: PARtonic Tomography Of Nucleon Software: A computing framework for the phenomenology of Generalized Parton Distributions. *Eur. Phys. J. C* 78.6 (2018), p. 478. DOI: 10.1140/epjc/s10052-018-5948-0.

[50] E. C. Aschenauer et al. EpIC: novel Monte Carlo generator for exclusive processes. *Eur. Phys. J. C* 82.9 (2022), p. 819. DOI: 10.1140/epjc/s10052-022-10651-z.

[51] PARTONS web page. http://partons.cea.fr.

[52] EpIC repository. https://pawelsznajder.github.io/epic/.

[53] S. Goloskokov and P. Kroll. The longitudinal cross section of vector meson electroproduction. *Eur. Phys. J. C* 50.4 (Mar. 2007), pp. 829–842. DOI: 10.1140/epjc/s10052-007-0228-4.

[54] S. Goloskokov and P. Kroll. The role of the quark and gluon GPDs in hard vector-meson electroproduction. *Eur. Phys. J. C* 53.3 (Nov. 2007), pp. 367–384. DOI: 10.1140/epjc/s10052-007-0466-5.

[55] M. Vanderhaeghen, P. A. M. Guichon, and M. Guidal. Hard Electroproduction of Photons and Mesons on the Nucleon. *Phys. Rev. Lett.* 80 (23 June 1998), pp. 5064–5067. DOI: 10.1103/PhysRevLett.80.5064.

[56] M. Vanderhaeghen, P. A. M. Guichon, and M. Guidal. Deeply virtual electroproduction of photons and mesons on the nucleon: Leading order amplitudes and power corrections. *Phys. Rev. D* 60 (9 Oct. 1999), p. 094017. DOI: 10.1103/PhysRevD.60.094017.

[57] K. Goeke, M. Polyakov, and M. Vanderhaeghen. Hard exclusive reactions and the structure of hadrons. *Prog. Part. Nucl. Phys.* 47.2 (Jan. 2001), pp. 401–515. DOI: 10.1016/s0146-6410(01)00158-2.

[58] M. Guidal et al. Nucleon form factors from generalized parton distributions. *Phys. Rev. D* 72 (5 Sept. 2005), p. 054013. DOI: 10.1103/PhysRevD.72.054013.

[59] C. Mezrag, H. Moutarde, and F. Sabatié. Test of two new parametrizations of the generalized parton distribution H. *Phys. Rev. D* 88.1 (2013), p. 014001. DOI: 10.1103/PhysRevD.88.014001.

[60] V. Bertone. APFEL++: A new PDF evolution library in C++. *PoS* DIS2017 (2018). Ed. by U. Klein, p. 201. DOI: 10.22323/1.297.0201.

[61] L. W. Mo and Y.-S. Tsai. Radiative Corrections to Elastic and Inelastic e p and mu p Scattering. *Rev. Mod. Phys.* 41 (1969), pp. 205–235. DOI: 10.1103/RevModPhys.41.205.

[62] J. Kripfganz, H. J. Möhring, and H. Spiesberger. Higher order leading logarithmic QED corrections to deep inelastic ep scattering at very high energies. *Z. Phys. C* 49 (1991), pp. 501–510. DOI: 10.1007/BF01549704.

[63] S. Jadach and P. Sawicki. mFOAM-1.02: A compact version of the cellular event generator FOAM. *Computer Physics Communications* 177.5 (2007), pp. 441–458. ISSN: 0010-4655. DOI: https://doi.org/10.1016/j.cpc.2007.02.112.

[64] R. Kleiss. The Cross-section for $e^+e^- \to e^+e^-e^+e^-$. *Nucl. Phys. B* 241 (1984), p. 61. DOI: 10.1016/0550-3213(84)90197-4.

[65] R. Kleiss and W. J. Stirling. Spinor Techniques for Calculating p anti-p —> W+- / Z0 + Jets. *Nucl. Phys. B* 262 (1985), pp. 235–262. DOI: 10.1016/0550-3213(85)90285-8.

[66] V. M. Braun, A. N. Manashov, and B. Pirnay. Finite-t and target mass corrections to DVCS on a scalar target. *Phys. Rev. D* 86 (2012), p. 014003. DOI: 10.1103/PhysRevD.86.014003.

[67] V. M. Braun, A. N. Manashov, and B. Pirnay. Finite-t and target mass corrections to deeply virtual Compton scattering. *Phys. Rev. Lett.* 109 (2012), p. 242001. DOI: 10.1103/PhysRevLett.109.242001.

[68] V. M. Braun et al. Deeply Virtual Compton Scattering to the twist-four accuracy: Impact of finite-t and target mass corrections. *Phys. Rev. D* 89.7 (2014), p. 074022. DOI: 10.1103/PhysRevD.89.074022.

[69] K. Deja et al. Phenomenology of double deeply virtual Compton scattering in the era of new experiments. *Phys. Rev. D* 107.9 (2023), p. 094035. DOI: 10.1103/PhysRevD.107.094035.

[70] K. Deja et al. Prospects for GPDs Extraction with Double DVCS. *Acta Phys. Polon. Supp.* 16.7 (2023), 7–A24. DOI: 10.5506/APhysPolBSupp.16.7-A24.

[71] K. Deja et al. Double DVCS as a window to the complete mapping of GPDs. *30th International Workshop on Deep-Inelastic Scattering and Related Subjects*. May 2023. arXiv: 2305.04274 [hep-ph].

[72] K. Deja et al. Can we measure Double DVCS at JLab and the EIC? *25th International Spin Symposium*. Dec. 2023. arXiv: 2401.13064 [hep-ph].

[73] V. Martínez-Fernández. Double DVCS including kinematic twist-4 corrections. 31st International Workshop on Deep Inelastic Scattering (DIS2024). Apr. 2024. URL: https://lpsc-indico.in2p3.fr/event/3268/contributions/7481/.

[74] A. V. Belitsky, D. Mueller, and A. Kirchner. Theory of deeply virtual Compton scattering on the nucleon. *Nucl. Phys. B* 629 (2002), pp. 323–392. DOI: 10.1016/S0550-3213(02)00144-X.

[75] A. Bacchetta et al. Single-spin asymmetries: The Trento conventions. *Phys. Rev. D* 70 (11 Dec. 2004), p. 117504. DOI: 10.1103/PhysRevD.70.117504.

[76] A. V. Belitsky and D. Müller. Exclusive electroproduction of lepton pairs as a probe of nucleon structure. *Phys. Rev. Lett.* 90 (2003), p. 022001. DOI: 10.1103/PhysRevLett.90.022001.

[77] R. Kleiss. The Cross-section for $e^+e^- \to e^+e^-e^+e^-$. *Nucl. Phys. B* 241 (1984), p. 61. DOI: 10.1016/0550-3213(84)90197-4.

[78] R. Kleiss and W. J. Stirling. Spinor Techniques for Calculating p anti-p —> W+- / Z0 + Jets. *Nucl. Phys. B* 262 (1985), pp. 235–262. DOI: 10.1016/0550-3213(85)90285-8.

[79] I. V. Anikin, B. Pire, and O. V. Teryaev. On the gauge invariance of the DVCS amplitude. *Phys. Rev. D* 62 (2000), p. 071501. DOI: 10.1103/PhysRevD.62.071501.

[80] M. Vanderhaeghen, P. A. M. Guichon, and M. Guidal. Deeply virtual electroproduction of photons and mesons on the nucleon: Leading order amplitudes and power corrections. *Phys. Rev. D* 60 (1999), p. 094017. DOI: 10.1103/PhysRevD.60.094017.

[81] A. Belitsky and D. Müller. Twist-three effects in two-photon processes. *Nucl. Phys. B* 589.3 (Nov. 2000), pp. 611–630. DOI: 10.1016/s0550-3213(00)00542-3.

[82] P. Kessler. *Workshop on Two-Photon Physics at LEP and HERA, Lund (Sweden)*. Photon fluxes and the EPA. May 1994.

[83] P. Kessler. IV. Théorie / Theory. The equivalent photon approximation in one- and two-photon exchange processes. *Journal de Physique Colloques* 35 (C2) (1974), pp. C2-97–C2-107. DOI: 10.1051/jphyscol:1974213.

[84] A. Belitsky, D. Müller, and Y. Ji. Compton scattering: From deeply virtual to quasi-real. *Nucl. Phys. B* 878 (2014), pp. 214–268. ISSN: 0550-3213. DOI: https://doi.org/10.1016/j.nuclphysb.2013.11.014.

[85] A. V. Radyushkin. Generalized Parton Distributions and Their Singularities. *Phys. Rev. D* 83 (2011), p. 076006. DOI: 10.1103/PhysRevD.83.076006.

[86] A. V. Radyushkin. Double distributions and evolution equations. *Phys. Rev. D* 59 (1999), p. 014030. DOI: 10.1103/PhysRevD.59.014030.

[87] P. Chatagnon et al. First Measurement of Timelike Compton Scattering. *Phys. Rev. Lett.* 127.26 (2021), p. 262501. DOI: 10.1103/PhysRevLett.127.262501.

[88] O. Grocholski et al. Data-driven study of timelike Compton scattering. *Eur. Phys. J. C* 80.2 (2020), p. 171. DOI: 10.1140/epjc/s10052-020-7700-9.

[89] V. M. Braun et al. Two-loop conformal generators for leading-twist operators in QCD. *JHEP* 03 (2016), p. 142. DOI: 10.1007/JHEP03(2016)142.

[90] K. G. Wilson. Non-Lagrangian Models of Current Algebra. *Phys. Rev.* 179 (1969), pp. 1499–1512. DOI: 10.1103/PhysRev.179.1499.

[91] D. J. Gross and S. B. Treiman. Light cone structure of current commutators in the gluon quark model. *Phys. Rev. D* 4 (1971), pp. 1059–1072. DOI: 10.1103/PhysRevD.4.1059.

[92] J. R. Ellis and R. L. Jaffe. Scaling, Short Distances and the Light Cone. SLAC-PUB-1353. Lectures presented at the University of California Santa Cruz Summer School on Particle Physics, June 25–July 6. 1973.

[93] B. Geyer, M. Lazar, and D. Robaschik. Decomposition of nonlocal light cone operators into harmonic operators of definite twist. *Nucl. Phys. B* 559 (1999), pp. 339–377. DOI: 10.1016/S0550-3213(99)00334-X.

[94] M. Srednicki. *Quantum field theory*. Cambridge University Press, 2007. ISBN: 978-0-521-86449-7, 978-0-511-26720-8.

[95] V. M. Braun, G. P. Korchemsky, and D. Müller. The Uses of conformal symmetry in QCD. *Prog. Part. Nucl. Phys.* 51 (2003), pp. 311–398. DOI: 10.1016/S0146-6410(03)90004-4.

[96] V. M. Braun et al. Two-loop evolution equations for flavor-singlet light-ray operators. *JHEP* 02 (2019), p. 191. DOI: 10.1007/JHEP02(2019)191.

[97] The renormalization group and the ϵ expansion. *Physics Reports* 12.2 (1974), pp. 75–199. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(74)90023-4.

[98] A. V. Belitsky and A. V. Radyushkin. Unraveling hadron structure with generalized parton distributions. *Phys. Rept.* 418 (2005), pp. 1–387. DOI: 10.1016/j.physrep.2005.06.002.

[99] J. D. Qualls. Lectures on Conformal Field Theory. Nov. 2015. arXiv: hep-th/1511.04074 [hep-th].

[100] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997. ISBN: 978-0-387-94785-3, 978-1-4612-7475-9. DOI: 10.1007/978-1-4612-2256-9.

[101] L. Fernando-Alday. Conformal Field Theory. https://courses.maths.ox.ac.uk/pluginfile.php/24397/mod_resource/content/1/CFTlectures.pdf (Lecture notes).

[102] H. Osborn. Lectures on Conformal Field Theories in more than two dimensions. https://www.damtp.cam.ac.uk/user/ho/CFTNotes.pdf.

[103] S. B. Treiman, R. Jackiw, and D. J. Gross. *Lectures on current algebra and its applications*. Princeton University Press, 1972. ISBN: 0-691-08118-2.

[104] D. Simmons-Duffin. The Conformal Bootstrap. *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings*. 2017, pp. 1–74. DOI: 10.1142/9789813149441_0001.

[105] S. Ferrara et al. The shadow operator formalism for conformal algebra. Vacuum expectation values and operator products. *Lett. Nuovo Cim.* 4S2 (1972), pp. 115–120. DOI: 10.1007/BF02907130.

[106] R. Jackiw and S. .-Y. Pi. Tutorial on Scale and Conformal Symmetries in Diverse Dimensions. *J. Phys. A* 44 (2011), p. 223001. DOI: 10.1088/1751-8113/44/22/223001.

[107] S. Ferrara, A. F. Grillo, and R. Gatto. Tensor representations of conformal algebra and conformally covariant operator product expansion. *Annals Phys.* 76 (1973), pp. 161–188. DOI: 10.1016/0003-4916(73)90446-6.

[108] S. Ferrara et al. Canonical scaling and conformal invariance. *Physics Letters B* 38.5 (1972), pp. 333–334.

[109] S. Ferrara and G. Parisi. Conformal covariant correlation functions. *Nuclear physics B* 42 (1972), pp. 281–290.

[110] S. Ferrara, A. F. Grillo, and R. Gatto. Manifestly conformal covariant operator-product expansion. *Lett. Nuovo Cim.* 2S2 (1971), pp. 1363–1369. DOI: 10.1007/BF02770435.

[111] J. A. M. Vermaseren, A. Vogt, and S. Moch. The Third-order QCD corrections to deep-inelastic scattering by photon exchange. *Nucl. Phys. B* 724 (2005), pp. 3–182. DOI: 10.1016/j.nuclphysb.2005.06.020.

[112] M. V. Polyakov and C. Weiss. Skewed and double distributions in pion and nucleon. *Phys. Rev. D* 60 (1999), p. 114017. DOI: 10.1103/PhysRevD.60.114017.

[113] R. Tarrach. Invariant Amplitudes for Virtual Compton Scattering Off Polarized Nucleons Free from Kinematical Singularities, Zeros and Constraints. *Nuovo Cim. A* 28 (1975), p. 409. DOI: 10.1007/BF02894857.

[114] S. Fucini, S. Scopetta, and M. Viviani. Coherent deeply virtual Compton scattering off ^{4}He. *Phys. Rev. C* 98 (1 July 2018), p. 015203. DOI: 10.1103/PhysRevC.98.015203.

[115] D. Amrath, M. Diehl, and J. Lansberg. Deeply virtual Compton scattering on a virtual pion target. *Eur. Phys. J. C* 58 (Oct. 2008), pp. 179–192. DOI: 10.1140/epjc/s10052-008-0769-1.

[116] J. M. M. Chávez et al. Accessing the Pion 3D Structure at US and China Electron-Ion Colliders. *Phys. Rev. Lett.* 128.20 (2022), p. 202501. DOI: 10.1103/PhysRevLett.128.202501.

[117] J. M. M. Chávez et al. Pion generalized parton distributions: A path toward phenomenology. *Phys. Rev. D* 105 (9 May 2022), p. 094012. DOI: 10.1103/PhysRevD.105.094012.

[118] J. D. Sullivan. One-Pion Exchange and Deep-Inelastic Electron-Nucleon Scattering. *Phys. Rev. D* 5 (7 1972), pp. 1732–1737. DOI: 10.1103/PhysRevD.5.1732.

[119] C. Lorcé, B. Pire, and Q.-T. Song. Kinematical higher-twist corrections in $\gamma^*\gamma \to M\overline{M}$. *Phys. Rev. D* 106 (9 Nov. 2022), p. 094030. DOI: 10.1103/PhysRevD.106.094030.

[120] B. Pire and Q.-T. Song. Kinematical higher-twist corrections in $\gamma^* \to M_1 M_2 \gamma$: Neutral meson production. *Phys. Rev. D* 107 (11 June 2023), p. 114014. DOI: 10.1103/PhysRevD.107.114014.

[121] M. Diehl. Generalized parton distributions. *Phys. Rept.* 388 (2003), pp. 41–277. DOI: 10.1016/j.physrep.2003.08.002.

[122] D. Zagier. The Dilogarithm Function. *Frontiers in number theory, physics, and geometry II, 3-65 (2007)* (Jan. 2007). DOI: 10.1007/978-3-540-30308-4_1.

[123] A. V. Belitsky and D. Müller. Off forward gluonometry. *Phys. Lett. B* 486 (2000), pp. 369–377. DOI: 10.1016/S0370-2693(00)00773-5.

[124] J. M. M. Chavez et al. Pion generalized parton distributions: A path toward phenomenology. *Phys. Rev. D* 105.9 (2022), p. 094012. DOI: 10.1103/PhysRevD.105.094012.

[125] A. V. Radyushkin. Generalized parton distributions (Oct. 2000). DOI: 10.1142/9789812810458_0027.

[126] D. Müller et al. On timelike and spacelike hard exclusive reactions. *Phys. Rev. D* 86 (2012), p. 031502. DOI: 10.1103/PhysRevD.86.031502.

[127] C. F. Perdrisat, V. Punjabi, and M. Vanderhaeghen. Nucleon Electromagnetic Form Factors. *Prog. Part. Nucl. Phys.* 59 (2007), pp. 694–764. DOI: 10.1016/j.ppnp.2007.05.001.

[128] R. L. Workman et al. Review of Particle Physics. *PTEP* 2022 (2022), p. 083C01. DOI: 10.1093/ptep/ptac097.

[129] N. J. Vilekin and A. U. Klimyk. *Representation of Lie groups and special functions*. Kluwer Academic Publishers, 1993. ISBN: 0-7923-1492-1, 0-7923-1494-8.

[130] M. Fierz and W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proc. Roy. Soc. Lond. A* 173 (1939), pp. 211–232. DOI: 10.1098/rspa.1939.0140.

[131] W. Rarita and J. Schwinger. On a theory of particles with half integral spin. *Phys. Rev.* 60 (1941), p. 61. DOI: 10.1103/PhysRev.60.61.

[132] T. R. Hemmert, B. R. Holstein, and J. Kambor. Heavy baryon chiral perturbation theory with light deltas. *J. Phys. G* 24 (1998), pp. 1831–1859. DOI: 10.1088/0954-3899/24/10/003.

[133] F. J. Milford. Projection Operator for the Rarita-Schwinger Equation. *Phys. Rev.* 98 (1955), pp. 1488–1488. DOI: 10.1103/PhysRev.98.1488.

[134] H. M. Siahaan. Spin-3/2 Field Theories and The Gauge Invariance. M.S. thesis at Technological Institute of Bandung, available at http://www.fisikanet.lipi.go.id/data/1014224401/data/1202810210.pdf. 2008.

[135] J. Estévez et al. Lightest flavor-singlet *qqq* baryons as witnesses to color. *Phys. Rev. D* 102.11 (2020), p. 114032. DOI: 10.13140/RG.2.2.36673.38244.

[136] V. M. Braun, A. N. Manashov, and J. Rohrwild. Baryon Operators of Higher Twist in QCD and Nucleon Distribution Amplitudes. *Nucl. Phys. B* 807 (2009), pp. 89–137. DOI: 10.1016/j.nuclphysb.2008.08.012.

[137] B. M. Pirnay. Higher twist effects in deeply virtual Compton scattering. PhD thesis. Regensburg U., 2016.

A

“Speed” of a virtual photon

Let us consider a process where the emission of a virtual photon happens. With respect to some reference frame ($\mathcal{O}$), its momentum may be written as $q^\mu = (\omega, \omega\vec{v})$, where ω is a real parameter and $\vec{v}$ is a vector in the direction of the photon motion. Also, let us consider the photon momentum to be timelike, this is $q^2 = Q^2 > 0$. Hence, one can prove that $\omega = Q/\sqrt{1-v^2}$ with $v = |\vec{v}|$.

In the reference frame of the photon ($\mathcal{O}^\star$) where it has zero three-momentum, we can write $q^{\star\,\mu} = (Q, \vec{0})$. Note that $\star$ indicates momentum with respect to $\mathcal{O}^\star$.

Now we will show that the vector $\vec{v}$ can be considered as the “speed” of the virtual photon since the the relative velocity $v_{\rm rel}$ of the frame $\mathcal{O}^\star$ with respect to $\mathcal{O}$ is precisely $v = |\vec{v}|$.

The boost from $\mathcal{O}$ to $\mathcal{O}^\star$ is along the direction of $\vec{v}$, so let us take $\vec{v} = (0, 0, v)$. This way, the Lorentz transformation is in the $z-$direction and

$$q^\star = \begin{pmatrix} Q \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(q^0 - v_{\rm rel} q^3) \\ 0 \\ 0 \\ \gamma(q^3 - v_{\rm rel} q^0) \end{pmatrix} \Rightarrow \begin{cases} Q = \gamma\omega(1 - v_{\rm rel} v) \\ 0 = \gamma\omega(v - v_{\rm rel}) \end{cases} \Rightarrow v_{\rm rel} = v\,,$$

where it was defined $\gamma = 1/\sqrt{1-v^2}$.

In this sense, v is the “speed” of the virtual photon and the rapidity is $\zeta = \operatorname{arctanh} v$. This rapidity is used in chapter 2 to transform the momenta of muon-antimuon system produced by the virtual photon of the reaction (2.1) and described in their center of mass frame back to the so-called target rest frame II (TRF-II).

B

Decomposition of hadron momenta by lightlike vectors

In order to use Eqs. (2.59) to (2.62) to describe the hadron spinors in (2.79) and (2.80), we need to find some lightlike vectors $\{r_1, r_2, r'_1, r'_2\}$ such that $p = r_1 + r_2$ and $p' = r'_1 + r'_2$. In this appendix we show a possible choice, in particular the one implemented in PARTONS. Remember that our formulation in Ch. 2 is completely independent of this choice.

Let us first introduce two lightlike vectors in TRF-II frame,

$$n_1^\mu = N_1(1,0,0,1)\,, \quad n_2^\mu = N_2(1,0,0,-1)\,, \quad N_1,\, N_2 \in \mathbb{R}\,, \tag{B.1}$$

that satisfy $n_1 n_2 = 1$, where $N_2 = 1/(2N_1)$. For simplicity, $N_1 = N$ that we define by imposing $n_2 \bar{p} = 1$.

In the rest frame of the hadron, for its momentum before interaction

$$p^\mu = (M, \vec{0}) = \frac{M}{2N} n_1^\mu + NM n_2^\mu\,, \tag{B.2}$$

and it is straightforward to identify

$$r_1^\mu = \frac{M}{2N} n_1^\mu\,, \quad r_2^\mu = NM n_2^\mu\,. \tag{B.3}$$

For the momentum of the scattered hadron,

$$p'^\mu = \alpha n_1^\mu + \beta n_2^\mu + p'^\mu_\perp = r'^\mu_1 + r'^\mu_2\,, \quad \alpha,\, \beta \in \mathbb{R}\,, \tag{B.4}$$

where

$$r'^\mu_1 = (\alpha - \gamma) n_1^\mu, \quad r'^\mu_2 = \gamma n_1^\mu + \beta n_2^\mu + p'^\mu_\perp\,, \quad p'^2_\perp < 0\,. \tag{B.5}$$

The coefficient $\gamma \in \mathbb{R}$ is introduced in order to make r'_2 lightlike. Coefficients α and β are trivially obtained by projecting p' with n_2 and n_1, respectively. Consequently, γ can be expressed by means of α and β as

$$\left.\begin{aligned} p'^2 &= M^2 \\ r'^2_2 &= 0 \end{aligned}\right\} \Rightarrow \gamma = \alpha - \frac{M^2}{2\beta}\,. \tag{B.6}$$

C

Parameterization of Dirac and Pauli electromagnetic form factors

Sachs' elastic form factors in terms of the Dirac's (F_1) and Pauli's (F_2) are given by

$$G_E(t) = F_1(t) + \frac{t}{4M^2} F_2(t)\,, \quad G_M(t) = F_1(t) + F_2(t)\,, \tag{C.1}$$

where M is the hadron mass and t the momentum carried by the photon striking the hadron. Conversely,

$$F_1(t) = \frac{4M^2 G_E(t) - t\, G_M(t)}{4M^2 - t}\,, \tag{C.2}$$

$$F_2(t) = \frac{4M^2 \left(G_M(t) - G_E(t)\right)}{4M^2 - t}\,. \tag{C.3}$$

Sachs' FFs can be parameterized via a dipole FF, F_D cf. [127], as

$$G_E(t) = F_D(t)\,, \quad G_M(t) = \mu F_D(t)\,, \tag{C.4}$$

where μ is the magnetic moment of the hadron: $\mu = 2.7928$ for proton and -1.9130 for neutron [128]. The dipole FF, F_D, is

$$F_D(t) = \left(1 - \frac{t}{\Lambda^2}\right)^{-2}\,, \tag{C.5}$$

with dipole mass $\Lambda^2 = 0.71$ GeV^2 for both proton and neutron. This parameterization for the EFFs is the one used in the DDVCS modules of `PARTONS`.

D

Projector onto geometric LT for (pseudo-)scalar operators

Scalar and pseudo-scalar operators built out of quark fields are of the form

$$\mathcal{O}(\lambda_1 z, \lambda_2 z) = (\lambda_2 - \lambda_1)\bar{\mathfrak{q}}(\lambda_1 z)\Gamma\mathcal{W}(\lambda_1 z, \lambda_2 z)\mathfrak{q}(\lambda_2 z)\,, \quad \lambda_i \in \mathbb{R}\,, \tag{D.1}$$

where the Γ-structure can be of different types such as $\Gamma = \{1, \not{z}, z^\mu \sigma_{\mu\nu}\partial^\nu, \gamma^5\not{z}, \gamma^5\}$ and $\mathcal{W}$ is the usual Wilson line, which ensures the gauge invariance of the operator above.

Because in Taylor expansion the covariant derivatives are multiplied by $z^{\mu_1}\cdots z^{\mu_n}$, the possible Lorentz representations of the local operators will be those of $D_{(\mu_1}\cdots D_{\mu_n)}$. Here, $(\cdots)$ represents total symmetrization and division by $n!$.

The symmetric product of covariant derivatives allows for the following Clebsch-Gordan decomposition in irreducible representations (irreps) of the Lorentz group:

$$D_{(\mu_1}\cdots D_{\mu_n)} \sim \left(\frac{n}{2}, \frac{n}{2}\right) \oplus \left(\frac{n-2}{2}, \frac{n-2}{2}\right) \oplus \left(\frac{n-4}{2}, \frac{n-4}{2}\right) \oplus \cdots\,. \tag{D.2}$$

The representation $(n/2, n/2)$ carries the symmetric and traceless tensors, whilst the others the traces with increasing number of metric tensors.

Hence, the highest spin is n and the local operators with $\Gamma = \{1, \not{z}, z^\mu \sigma_{\mu\nu}\partial^\nu, \gamma^5\not{z}, \gamma^5\}$ have energy dimension $d = \{n+3, n+2, n+3, n+2, n+3\}$, implying that the lowest geometric twist is $\tau_{\min} = \{3, 2, 3, 2, 3\}$, respectively. In this appendix twist refers to its geometric version, this is "dimension of operator minus spin."

To show how we can obtain the twist decomposition of one of this operators, let us work out the particular example

$$\mathcal{O}(0, \lambda z) = \bar{\mathfrak{q}}(0)\lambda\not{z}\mathcal{W}(0, \lambda z)\mathfrak{q}(\lambda z)\,, \tag{D.3}$$

whose Taylor decomposition reads

$$\begin{aligned}\mathcal{O}(0,\lambda z) &= \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} z^{\beta} z^{\mu_1} \cdots z^{\mu_n} \bar{\mathfrak{q}}(0)\gamma_{\beta} D_{\mu_1} \cdots D_{\mu_n} \mathfrak{q}(0) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} \bar{\mathfrak{q}}(0) \not{z} (zD)^n \mathfrak{q}(0) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} \mathcal{O}^{(n+1)}(z)\,. \end{aligned} \tag{D.4}$$

Here, it has been defined

$$z^{\mu_1} \cdots z^{\mu_n} D_{\mu_1} \cdots D_{\mu_n} \equiv (zD)^n \tag{D.5}$$

and

$$\mathcal{O}^{(n+1)}(z) = \bar{\mathfrak{q}}(0) \not{z} (zD)^n \mathfrak{q}(0)\,. \tag{D.6}$$

Since γ_β is contracted with z^β, index β is subject to symmetrization with the other ones $\mu_1, ..., \mu_n$. Because all indices $\beta, \mu_1, ..., \mu_n$ are fully symmetrized, then we have the following Young tableau:

$$\boxed{\beta}\boxed{\mu_1}\boxed{\mu_2}\boxed{\cdots}\boxed{\mu_n} \quad\longmapsto\quad \mathcal{O}^{(n+1)}_{\beta\mu_1\cdots\mu_n} = \bar{\mathfrak{q}}(0)\gamma_{(\beta} D_{\mu_1} \cdots D_{\mu_n)} \mathfrak{q}(0)\,. \tag{D.7}$$

Now we want operators $\mathcal{O}^{(n+1)}_{\beta\mu_1\cdots\mu_n}$ to be made traceless. A totally symmetric and traceless tensor of rank n , $\mathring{T}^{(n)}_{\mu_1\cdots\mu_n}$, must satisfy after contraction with $z^{\mu_1} \cdots z^{\mu_n}$ the 4-dimensional Laplace equation,

$$\partial^2 \mathring{T}^{(n)}(z) = 0\,, \quad \text{with} \quad \mathring{T}^{(n)}(z) = z^{\mu_1} \cdots z^{\mu_n} \mathring{T}^{(n)}_{\mu_1\cdots\mu_n}\,. \tag{D.8}$$

Indeed, $\partial^2(z^{\mu_i} z^{\mu_j}) = 2g^{\mu_i\mu_j}$ so when applied to $\mathring{T}^{(n)}$ it renders

$$\partial^2 \mathring{T}^{(n)}(z) = 2(z^{\mu_3} \cdots z^{\mu_n} \mathring{T}^{(n)\Lambda}_{\Lambda\mu_3\cdots\mu_n} + z^{\mu_2} z^{\mu_4} \cdots z^{\mu_n} \mathring{T}^{(n)\Lambda}_{\mu_2\Lambda\mu_4\cdots\mu_n} + \cdots) = 0\,. \tag{D.9}$$

Since all zs are arbitrary, this last expression can only be true if all traces with two contracted indices are zero

$$\mathring{T}^{(n)\Lambda}_{\Lambda\mu_3\cdots\mu_n} = 0\,, \quad \mathring{T}^{(n)\Lambda}_{\mu_2\Lambda\mu_4\cdots\mu_n} = 0\,, \quad \cdots\,. \tag{D.10}$$

However, not only traces for two indices must be zero but all traces with any number of pairs of indices contracted. Fortunately, satisfying the four-dimensional Laplace equation ensures it:

$$\partial^2 \mathring{T}^{(n)}(z) = 0 \Rightarrow (\partial^2)^m \mathring{T}^{(n)}(z) = 0 \quad \text{with} \quad 1 \le m \le \left[\frac{n}{2}\right]\,. \tag{D.11}$$

Hereafter $\left[\frac{n}{2}\right]$ stands for the integral part of $\frac{n}{2}$.

According to [93] and chapter 9, section 9.2.3, in [129], the solutions for Eq. (D.8) for a D-dimensional space are the so-called *harmonic tensor functions* that read

$$\begin{aligned}\mathring{T}^{(n)}(z) &= \left\{1+\sum_{k=1}^{[n/2]}\left(\prod_{\ell=1}^{k}\frac{1}{D+2n-2\ell-2}\right)\frac{(-z^2)^k(\partial^2)^k}{2^k k!}\right\}T^{(n)}(z)\\ &= H^{(n,D)}(z^2|\partial^2)T^{(n)}(z)\,. \end{aligned} \tag{D.12}$$

In particular, for $D=4$

$$\begin{aligned}\mathring{T}^{(n)}(z) &= \sum_{k=0}^{[n/2]}\left(\prod_{\ell=1}^{k}\frac{1}{n-\ell+1}\right)\frac{(-z^2)^k(\partial^2)^k}{4^k k!}T^{(n)}(z)\\ &= \sum_{k=0}^{[n/2]}\frac{(n-k)!}{k!n!}\left(\frac{-z^2}{4}\right)^k(\partial^2)^k T^{(n)}(z)\,, \end{aligned} \tag{D.13}$$

where it was used

$$\prod_{\ell=1}^{k}\frac{1}{n-\ell+1}=\frac{1}{n(n-1)(n-2)\cdots(n-(k-1))}=\frac{(n-k)!}{n!}\,. \tag{D.14}$$

From here, you can conclude that the local traceless operators in Eq. (D.4) can be written as

$$\mathring{\mathcal{O}}^{(n+1)}(z)=\bar{\mathfrak{q}}(0)\not{z}(zD)^n\mathfrak{q}(0)\Big|_{\text{traceless}}=\sum_{k=0}^{[\frac{n+1}{2}]}\frac{(n+1-k)!}{k!(n+1)!}\left(\frac{-z^2}{4}\right)^k(\partial^2)^k\mathcal{O}^{(n+1)}(z)\,. \tag{D.15}$$

Recalling that factorial, gamma and beta functions are all related

$$\mathscr{B}(n,m)=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}=\frac{(n-1)!(m-1)!}{(n+m-1)!}\,, \tag{D.16}$$

and that there exists an integral representation for such function

$$\mathscr{B}(n,m)=\int_0^1 dt\, t^{n-1}(1-t)^{m-1}\quad\text{if and only if}\quad\mathfrak{Re}(n)>0\text{ and }\mathfrak{Re}(m)>0\,, \tag{D.17}$$

we can re-write Eq. (D.15) where the beta function is $\mathscr{B}(n+2-k,\,k)$. For the integral to hold it must happen that $n+2>k>0$, so in the sum over k we have to isolate the term for $k=0$. Doing so,

$$\mathring{\mathcal{O}}^{(n+1)}(z)=\left[1+\sum_{k=1}^{\infty}\int_0^1 dt\left(\frac{-z^2}{4}\right)^k\frac{(\partial^2)^k t^n}{k!(k-1)!}\left(\frac{1-t}{t}\right)^{k-1}\right]\mathcal{O}^{(n+1)}(z)\,. \tag{D.18}$$

Sum over k has been left unbounded because d'Alembert operator acting k times on a polynomial of degree $n+1$ produces zero when $k>[(n+1)/2]$.

Now, we can introduce these traceless operators into Eq. (D.4) and obtain the following result

$$\begin{aligned}\mathring{\mathcal{O}}(0,\lambda z) &= \sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!}\mathcal{O}^{(n+1)}(z)\\ &\quad+\sum_{k=1}^{\infty}\int_0^1 dt\,\left(\frac{-z^2}{4}\right)^k\frac{(\partial^2)^k}{k!(k-1)!}\frac{(1-t)^{k-1}}{t^k}\sum_{n=0}^{\infty}\frac{(\lambda t)^{n+1}}{n!}\mathcal{O}^{(n+1)}(z)\\ &= \mathcal{O}(0,\lambda z)+\sum_{k=1}^{\infty}\int_0^1 dt\,\left(\frac{-z^2}{4}\right)^k\frac{(\partial^2)^k}{k!(k-1)!}\frac{(1-t)^{k-1}}{t^k}\mathcal{O}(0,\lambda tz)\,. \qquad \text{(D.19)}\end{aligned}$$

Because $\mathring{\mathcal{O}}$ is symmetric and traceless, it has the highest possible spin (n) and therefore represents the geometric leading twist (LT) component of the operator. Therefore,

$$\mathring{\mathcal{O}} = [\mathcal{O}]_{\mathrm{LT}}\,. \qquad \text{(D.20)}$$

Taking into account expansion in Eq. (D.4), the sum over k (that represent the traces) can be written explicitly by means of local tensors for which we can associate spin, dimension and twist,

$$\begin{aligned}&\sum_{k=1}^{\infty}\int_0^1 dt\,\left(\frac{-z^2}{4}\right)^k\frac{(\partial^2)^k}{k!(k-1)!}\frac{\bar{t}^{k-1}}{t^k}\mathcal{O}(0,\lambda tz) =\\ &\sum_{k=1}^{\infty}\int_0^1 dt\,\left(\frac{-z^2}{4}\right)^k\frac{(\partial^2)^k}{k!(k-1)!}\frac{(\bar{t})^{k-1}}{t^k}\sum_{n=0}^{\infty}\frac{\lambda t^{n+1}}{n!}z^\beta z^{\mu_1}\cdots z^{\mu_n}\bar{\mathfrak{q}}(0)\gamma_\beta D_{\mu_1}\cdots D_{\mu_n}\mathfrak{q}(0) =\\ &\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!}\left(\frac{-z^2}{4}\right)^k\frac{1}{k!(k-1)!}\int_0^1 dt\,\frac{\bar{t}^{k-1}}{t^k}t^{n+1}(\partial^2)^k x^\beta z^{\mu_1}\cdots z^{\mu_n}\bar{\mathfrak{q}}(0)\gamma_\beta D_{\mu_1}\cdots D_{\mu_n}\mathfrak{q}(0) =\\ &\sum_{k=1}^{\infty}\left[\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!}f^{(k,n)}(z^2)(\partial^2)^k z^\beta z^{\mu_1}\cdots z^{\mu_n}\bar{\mathfrak{q}}(0)\gamma_\beta D_{\mu_1}\cdots D_{\mu_n}\mathfrak{q}(0)\right]. \qquad \text{(D.21)}\end{aligned}$$

To highlight the tensor structure, we simplified the notation defining the function

$$f^{(k,n)}(z^2) = \left(\frac{-z^2}{4}\right)^k\frac{1}{k!(k-1)!}\int_0^1 dt\,\frac{\bar{t}^{k-1}}{t^k}t^{n+1}\,,\quad \bar{t}=1-t\,. \qquad \text{(D.22)}$$

From the last equation it is easy to realize that the action of $(\partial^2)^k$ generates a total of k metrics. Let us go order by order in k, focusing only on the tensor structure so that we can unveil the Lorentz irreps and so their twist.

For $k=1$,

$$\begin{aligned}\partial^2 z^\beta z^{\mu_1}\cdots z^{\mu_n}\gamma_\beta D_{\mu_1}\cdots D_{\mu_n} &\sim (g^{\beta\mu_1}z^{\mu_2}\cdots z^{\mu_n}+g^{\beta\mu_2}z^{\mu_1}z^{\mu_3}\cdots z^{\mu_n}+g^{\mu_1\mu_2}z^\beta z^{\mu_3}\cdots z^{\mu_n}\\ &\quad+g^{\mu_1\mu_3}z^\beta z^{\mu_2}z^{\mu_4}\cdots z^{\mu_n}+\cdots)\gamma_\beta D_{\mu_1}\cdots D_{\mu_n}\\ &\sim z^{\mu_2}\cdots z^{\mu_n}\big(\not{D}D_{\mu_2}\cdots D_{\mu_n}+D_{\mu_2}\not{D}D_{\mu_3}\cdots D_{\mu_n}+\cdots\\ &\quad+\gamma_{\mu_2}D^2D_{\mu_3}\cdots D_{\mu_n}+\gamma_{\mu_2}D^\lambda D_{\mu_3}D_\lambda D_{\mu_4}\cdots D_{\mu_n}+\cdots\big)\,.\end{aligned} \qquad \text{(D.23)}$$

The tensor structure in parenthesis represents that of the local trace operators. These ones are fully symmetrized under indices $\mu_2,...,\mu_n$, so the associated Young tableau is

$$\boxed{\mu_2}\boxed{\mu_3}\boxed{\cdots}\boxed{\mu_n}$$

corresponding to irrep $\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$, whose spin is $n-1$. Consequently traces for $k=1$ are geometric twist-4.

For $k>1$ you can find similar expressions with just more pairs of indices contracted, higher traces. For general k, you would have operators with spin $j_{n,k} = n+1-2k$ and energy dimension $d = 3+n$, so twist $\tau_k = 2+2k$. Consequently, the expansion in Eq. (D.19) is a geometric twist (τ) expansion:

$$\sum_{k=1}^{\infty} \to \sum_{\substack{\tau=4 \\ \text{Even } \tau}}^{\infty} .$$

Projecting onto the light-cone $z \to n$ ($n^2 = 0$) and taking into account that for $\tau \geq 4$ ($k \geq 1$) operators are multiplied by positive powers of $z^2 \to n^2 = 0$, we are left with

$$\mathring{\mathcal{O}}(0, \lambda n) = \mathcal{O}(0, \lambda n) = \bar{\mathfrak{q}}(0)(\gamma^\mu \lambda n_\mu)\mathcal{W}(0, \lambda n)\mathfrak{q}(\lambda n)\,, \tag{D.24}$$

which is already twist-2 and whose matrix elements describe the vector GPDs. In a similar manner, by considering $\gamma^\mu \to \gamma^\mu\gamma^5$ (pseudo-scalar operator) we get the axial-vector operators related to GPDs. This justifies why GPDs are usually referred to as distributions coming from geometric twist-2 operators.

E

Group generators and their representations

Let G be a generator of some group algebra that transforms an operator $\mathcal{O}^i_{(\mathcal{R})}$ into $\mathcal{O}^{i\,\prime}_{(\mathcal{R})}$ at a fixed spacetime point z:

$$\delta_G \mathcal{O}^i_{(\mathcal{R})}(z) = \mathcal{O}^{i\,\prime}_{(\mathcal{R})}(z) - \mathcal{O}^i_{(\mathcal{R})}(z)\,. \tag{E.1}$$

Here $\mathcal{R}$ is the representation of the group under which the operator transforms. Then, for every operator $\mathcal{O}^i_{(\mathcal{R})}$ (at a point z, considered implicit hereafter), the finite version of a symmetry transformation is

$$\mathcal{O}^{i\,\prime}_{(\mathcal{R})} = g\mathcal{O}^i_{(\mathcal{R})}g^{-1} = \mathcal{R}(g^{-1})^i_j\mathcal{O}^j_{(\mathcal{R})} \tag{E.2}$$

where $\mathcal{R}(g)$ is the representation of the group element g under which $\mathcal{O}^i_{(\mathcal{R})}$ transforms. Let us prove that indeed this formula is in accordance with the group transformation, this is that the representation satisfies the same multiplication form. Take two consecutive transformation so if Eq. (E.2) is correct, then

$$(g_1g_2)\mathcal{O}^i_{(\mathcal{R})}(g_1g_2)^{-1} = \mathcal{R}((g_1g_2)^{-1})^i_j\mathcal{O}^j_{(\mathcal{R})} \tag{E.3}$$

which can also be expressed as

$$\begin{aligned}(g_1g_2)\mathcal{O}^i_{(\mathcal{R})}(g_1g_2)^{-1} &= g_1g_2\mathcal{O}^i_{(\mathcal{R})}g_2^{-1}g_1^{-1}\\ &= g_1\mathcal{R}(g_2^{-1})^i_j\mathcal{O}^j_{(\mathcal{R})}g_1^{-1}\\ &= \mathcal{R}(g_2^{-1})g_1\mathcal{O}^j_{(\mathcal{R})}g_1^{-1}\\ &= \mathcal{R}(g_2^{-1})^i_j\mathcal{R}(g_1^{-1})^j_k\mathcal{O}^k_{(\mathcal{R})}\,,\end{aligned} \tag{E.4}$$

From the result above:

$$\mathcal{R}(g_2^{-1}g_1^{-1}) = \mathcal{R}(g_2^{-1})\mathcal{R}(g_1^{-1}) \tag{E.5}$$

or, equivalently, $\mathcal{R}(g_1g_2) = \mathcal{R}(g_1)\mathcal{R}(g_2)$. In other words, $\mathcal{R}$ satisfies the same group multiplication law as g so it is indeed a representation. If in RHS of Eq. (E.2) you picked $\mathcal{R}(g)$ instead of $\mathcal{R}(g^{-1})$, you would obtain $\mathcal{R}(g_1 g_2) = \mathcal{R}(g_2)\mathcal{R}(g_1)$, thus inverting the order of the group multiplication law. We conclude that the finite version of a symmetry transformation must indeed be Eq. (E.2).

We can now relate this to infinitesimal transformations. Given the group element g_ϵ and the Wigner and Stone theorems:

$$g_\epsilon = \exp[-i\epsilon G]\,, \tag{E.6}$$

where $\epsilon \in \mathbb{R}$ is the group parameter that measures "how much" the operator changes. Therefore, the representation of the group element is:

$$\mathcal{R}(g_\epsilon) = \exp[-i\epsilon\mathcal{R}(G)] = \exp[-\delta_G]\,, \tag{E.7}$$

which in turn renders the representation of the generators:

$$\mathcal{R}(G) = -\frac{i}{\epsilon}\delta_G\,. \tag{E.8}$$

This way, the infinitesimal transformation is given by

$$\delta_G \mathcal{O}^i_{(\mathcal{R})} = -i\epsilon[G, \mathcal{O}^i_{(\mathcal{R})}]\,. \tag{E.9}$$

In physics we often consider the group transformations acting on both spacetime coordinates and fields. It is custom to consider "forward" transformations for coordinates,

$$z^\mu \to z'^\mu = \mathcal{R}(g_\epsilon) z^\mu, \tag{E.10}$$

while for fields and operators rule (E.4) is followed,

$$\Phi'(z) = \mathcal{R}(g_\epsilon^{-1})\Phi(z) = \mathcal{R}(g_{-\epsilon})\Phi(z) \tag{E.11}$$

The reason behind this convention is that for $\delta_\epsilon z^\mu = z'^\mu - z^\mu = (\mathcal{R}(g_\epsilon) - 1)z^\mu$ and a field such that $\Phi'(z') = \Phi(z)$, then

$$\Phi'(z) = \Phi(z(z')) = \Phi(z - \delta_\epsilon z) \Rightarrow \delta_\epsilon \Phi(z) = -\delta_\epsilon z^\mu \partial_\mu \Phi(z) = (\mathcal{R}(g_{-\epsilon}) - 1)\Phi(z) \tag{E.12}$$

which makes sense since the condition $\Phi'(z') = \Phi(z)$ can be translated to $\Phi'(z) = \Phi(z - \delta_\epsilon z)$.

F

Inversion transformation

The inversion operation on fields cannot be found with the methods used thus far for two reasons: 1) they do not allow for exponential map, and 2) they modify the units of the object they act on, so the difference $(\mathcal{I}\Phi)(1/z) - \Phi(z)$ makes no sense. Hence, one can propose a series of *Ansätzes* for the inversion transformation of different spin-fields ensuring that the properties of $\mathcal{I}$ are satisfied. These *Ansätzes*, presented in [105, 106], can be proven to be the appropriate inversions by showing that Eq. (3.65) is indeed satisfied.

First, we are going to consider that an inversion, as any other transformation seen previously, can be written by means of a (matrix) representation[1] $\mathcal{R}(\mathcal{I})$

$$(\mathcal{I}\Phi)(1/z) = \mathcal{R}(\mathcal{I})\Phi(z)\,. \tag{F.1}$$

We claim:

- Inverse fields for being actual fields must vanish at infinite, hence $\mathcal{R}(\mathcal{I}) \propto (z^2)^{\mathscr{P}}$ with $\mathscr{P}$ some power. Also, by choosing z^2 as proportionality factor, $\mathcal{R}(\mathcal{I})$ does not map fields in a particular Lorentz irrep to other irreps.

- From Eq. (3.62), under re-scaling $z \to \lambda z$ the inverse coordinate transforms as $1/z \to \lambda^{-1}(1/z)$. This is, the behaviour of z and $1/z$ under scaling transformations is obviously the opposite of each other. This simple observation suggests that under dilations if Φ has scaling dimension ℓ, then $\mathcal{I}\Phi$ must transform with $\ell' = -\ell$. As a result, $\mathcal{R}(\mathcal{I}) = (z^2)^{\ell}\hat{\mathcal{R}}(\mathcal{I})$ with $\hat{\mathcal{R}}(\mathcal{I})$ invariant under re-scaling.

- Also, because $\ell' = -\ell$ inversions must satisfy $\mathcal{I}\hat{\boldsymbol{\ell}}\Phi(z) = -\hat{\boldsymbol{\ell}}\mathcal{I}\Phi(z)$ which translates to $\mathcal{I}\mathcal{D}\mathcal{I} = -\mathcal{D}$. Note that indeed $\mathcal{I}\hat{\boldsymbol{\ell}}\Phi(0) = -\hat{\boldsymbol{\ell}}\mathcal{I}\Phi(0)$ can be written as $\{\mathcal{I},\mathcal{D}\}\Phi(0) = 0$, and since (anti)commutation rules hold for any field then $\{\mathcal{I},\mathcal{D}\} = 0$.

- As inversions satisfy $\mathcal{I}^2 = 1$, then $\mathcal{R}(\mathcal{I}^2) = \mathcal{R}(\mathcal{I})^2 = (z^2)^{\ell+\ell'}\hat{\mathcal{R}}(\mathcal{I})^2 = \hat{\mathcal{R}}(\mathcal{I})^2 \equiv 1$ where the earlier result $\ell' = -\ell$ has been used.

- As stated before, $\mathcal{R}(\mathcal{I})$ must be chosen in such a way that relation (3.65) holds. This is enough to accept any *Ansätze* as the right inversions.

[1] Since inversions do not belong to the group algebra, by their "representation" we simply mean a matrix or operator that when acting on the field delivers the inverted field at the reversed position.

Now, our goal is to find the operators $\hat{\mathcal{R}}(\mathcal{I})$ for each spin-field attending these claims.

For the scalar field $\Phi = \phi$ we can take the simplest *Ansatz*, i.e., $\hat{\mathcal{R}}(\mathcal{I}) = 1$ so that

$$(\mathcal{I}\phi)(1/z) = (z^2)^{\ell}\phi(z)\,, \tag{F.2}$$

where ℓ is the scaling dimension of ϕ. Indeed, for $\mathcal{I}\phi$ the scaling dimension is the complementary $-\ell$. For a dilation $z \to \lambda z$, the dilated inverted field at the dilated reversed position is given by

$$(\mathcal{I}\phi)'(\lambda^{-1}/z) = \lambda^{2\ell}\lambda^{-\ell}\phi(z) = \lambda^{\ell}\phi(z)\,. \tag{F.3}$$

Comparing with Eq. (3.85) we can read out that the scaling dimension of $\mathcal{I}\phi$ is $-\ell$.

For vector fields $\Phi = A^{\rho}$ we can make use of Eq. (3.63) for a hint and take the inversion tensor (3.64) as *Ansatz* $\hat{\mathcal{R}}(\mathcal{I})^{\rho}_{\mu} = \mathcal{I}^{\rho}_{\mu}(z)$, so that

$$(\mathcal{I}A)^{\rho}(1/z) = (z^2)^{\ell}\mathcal{I}^{\rho}_{\mu}(z)A^{\mu}(z)\,, \tag{F.4}$$

where ℓ is the scaling dimension of $A^{\mu}(z)$. Property (3.68) ensures $\hat{\mathcal{R}}(\mathcal{I})^2 = 1$. With this *Ansatz*, that of tensors is immediate:

$$(\mathcal{I}\mathcal{O}_{\mu_1\mu_2\cdots\mu_n})(1/z) = (z^2)^{\ell}\mathcal{I}^{\nu_1}_{\mu_1}(z)\mathcal{I}^{\nu_2}_{\mu_2}(z)\cdots\mathcal{I}^{\nu_n}_{\mu_n}(z)\mathcal{O}_{\nu_1\nu_2\cdots\nu_n}(z) \tag{F.5}$$

and, as usual, ℓ is the scaling dimension of the tensor being transformed, $\mathcal{O}_{\mu_1\mu_2\cdots\mu_n}$.

For spin-1/2 fields $\Phi = \psi$, notice that $(\gamma\hat{z})^2 = 1$ which makes $\gamma\hat{z}$ a suitable candidate for $\hat{\mathcal{R}}(\mathcal{I})$. With ψ having scaling dimension ℓ:

$$(\mathcal{I}\psi)(1/z) = (z^2)^{\ell}(\gamma\hat{z})\psi(z)\,. \tag{F.6}$$

An extension of these formula for higher spins is also possible and needed to prove the validity of (F.6). These fields with spin $n + 1/2$ are known as *Fierz-Pauli spinors* [130] and lay on the Lorentz cover of

$$\underbrace{\left(\frac{1}{2},\frac{1}{2}\right)\otimes\cdots\otimes\left(\frac{1}{2},\frac{1}{2}\right)}_{n\text{ times, tensor}}\otimes\underbrace{\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]}_{\text{spin}-1/2\text{ field}}\,. \tag{F.7}$$

Objects belonging to these Lorentz representations can be denoted by $\Psi_{\mu_1\mu_2\cdots\mu_n}$. Each component of this tensor is a spinor satisfying Dirac equation. To pick up a precise spin field from the decomposition above, it is needed to find the projector onto the appropriate irrep. Spin-3/2 spinors have the alternative name of *Rarita-Schwinger spinors* and projectors onto irreducible spaces of spin-3/2 and spin-1/2 are well-known, cf. [131–135]. Hence, for Fierz-Pauli spinors with scaling dimension ℓ the inversion is given by

$$(\mathcal{I}\Psi_{\mu_1\mu_2\cdots\mu_n})(1/z) = (z^2)^{\ell}\mathcal{I}^{\nu_1}_{\mu_1}(z)\mathcal{I}^{\nu_2}_{\mu_2}(z)\cdots\mathcal{I}^{\nu_n}_{\mu_n}(z)(\gamma\hat{z})\Psi_{\nu_1\nu_2\cdots\nu_n}(z)\,. \tag{F.8}$$

Consequently, if an operator behaves under conformal transformations (including inversions) as shown in this section then it is called a conformal operator.

To test the validity of such expressions for the inversion transformation of fields, it is enough to check that relation (3.65) holds.

G

Conformal correlator for two scalar fields

For two scalar fields with, in principle, different scaling dimensions ℓ_1 and ℓ_2 for ϕ_1 and ϕ_2, respectively; in order for the correlator to be Poincaré invariant it must be

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = f(z_{12}^2)\,, \tag{G.1}$$

where we used $z_{12}^2 = (z_1 - z_2)^2$ and $f\colon \mathbb{R} \mapsto \mathbb{C}$ some function.
Applying a scale transformation $z \to \lambda z$ to Eq. (3.104), we have that

$$f(\lambda^2 z_{12}^2) = \lambda^{-(\ell_1+\ell_2)}\langle\phi_1(z_1)\phi_2(z_2)\rangle = \lambda^{-(\ell_1+\ell_2)} f(z_{12}^2) \tag{G.2}$$

and, as a consequence,

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \frac{c_{12}}{(z_{12}^2)^{(\ell_1+\ell_2)/2}}\,, \quad c_{12} \in \mathbb{C}\,. \tag{G.3}$$

We are left with special conformal transformation that by virtue of relation (3.65) we can substitute by the inversion operation. With Eqs. (3.104) and (G.3), for inversions we have:

$$\begin{aligned}\langle(\mathcal{I}\phi_1)(1/z_1)(\mathcal{I}\phi_2)(1/z_2)\rangle &= \langle\phi_1(1/z_1)\phi_2(1/z_2)\rangle\,;\\ (z_1^2)^{\ell_1}(z_2^2)^{\ell_2} f(z_{12}^2) &= f((1/z_1 - 1/z_2)^2)\,.\end{aligned} \tag{G.4}$$

Let us work out the term in RHS:

$$f((1/z_1 - 1/z_2)^2) = \frac{c_{12}}{\left(\left(\frac{z_1}{z_1^2} - \frac{z_2}{z_2^2}\right)^2\right)^{(\ell_1+\ell_2)/2}} = \frac{c_{12}}{\left(\frac{(z_1-z_2)^2}{z_1^2 z_2^2}\right)^{(\ell_1+\ell_2)/2}}\,. \tag{G.5}$$

that when introduced in Eq. (G.4) yields

$$(z_1^2)^{\ell_1}(z_2^2)^{\ell_2} f(x_{12}^2) = (z_1^2 \cdot z_2^2)^{(\ell_1+\ell_2)/2} f(z_{12}^2) \Leftrightarrow \ell_1 = \ell_2\,. \tag{G.6}$$

Therefore, we can conclude that the commutator of two scalar fields is given by ($z_{12} \neq 0$):

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \begin{cases} \dfrac{c_{12}}{(z_{12}^2)^{\ell_1}}\,, & \text{if } \ell_1 = \ell_2\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{G.7}$$

For $z_{12} = 0$, take into account that a Dirac delta in a D-dimensional spacetime satisfies

$$\delta(f(z)) = \sum_{\substack{z_i, \\ f(z_i)=0}} \frac{\delta(z - z_i)}{|\partial f(z_i)/\partial z|}\,, \tag{G.8}$$

where $\partial f(z_i)/\partial z = (\partial f(z)/\partial z)|_{z=z_i}$ is the Jacobian of f evaluated at z_i for z_i a zero of the function f. Under a dilation $z \to f(z) = \lambda z$ ($\lambda \in \mathbb{R}$) the Dirac delta transforms as

$$\delta(\lambda z) = \lambda^{-D}\delta(z)\,. \tag{G.9}$$

Therefore, considering the rest of the conformal transformations as explained above, for two scalar fields such that $\ell_1 + \ell_2 = D$ and $z_{12} = 0$, one concludes:

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = c_{12}\delta(z_{12})\,, \quad \ell_1 + \ell_2 = D\,. \tag{G.10}$$

H

Orthogonality of conformal $n-$ranked tensors

Equation (3.113) can be proven easily by considering the scalar functions [107, 108]

$$z_1^{\mu_1} \cdots z_1^{\mu_n} z_2^{\nu_1} \cdots z_2^{\nu_m} \langle \mathcal{O}_{1,\mu_1\cdots\mu_n}(z_1)\mathcal{O}_{2,\nu_1\cdots\nu_m}(z_2)\rangle = F(z_{12}^2)\,, \tag{H.1}$$

$$z_2^{\mu_1} \cdots z_2^{\mu_n} z_1^{\nu_1} \cdots z_1^{\nu_m} \langle \mathcal{O}_{1,\mu_1\cdots\mu_n}(z_1)\mathcal{O}_{2,\nu_1\cdots\nu_m}(z_2)\rangle = F'(z_{12}^2)\,. \tag{H.2}$$

Transformation properties of tensors and coordinates under dilations imply that $F(z_{12}^2)$ is a homogeneous function of degree $n - \ell_1$ in z_1 and of degree $m - \ell_2$ in z_2. Since F depends on the interval $(z_1 - z_2)^2$ (otherwise above equations would not represent scalars), for the above expression to be consistent it has to be

$$n - \ell_1 = m - \ell_2 \Rightarrow n - m = \ell_1 - \ell_2\,. \tag{H.3}$$

On an equal footing with $F(z_{12}^2)$, the function $F'(z_{12}^2)$ is homogeneous of degree $m - \ell_1$ in z_1 and of degree $n - \ell_2$ in z_2, so

$$m - \ell_1 = n - \ell_2 \Rightarrow -(n - m) = \ell_1 - \ell_2\,. \tag{H.4}$$

But for Eqs. (H.3) and (H.4) to be compatible, it must be $n = m$ and $\ell_1 = \ell_2$.

I

Light-cone coordinates in spinor formalism

The starting point of the spinor formalism is to realize that the vector representation of the Lorentz group $(1/2, 1/2)$ can be written as a product of spinor representations $(1/2, 0) \otimes (0, 1/2)$, i.e. these two representation are related and a mapping between them must exist. In what follows, we will untangle such mapping. For further details, cf. [66, 94, 136].

Let us represent a left-handed spinor field as ψ_a. Because this field belongs to the irreducible representation (irrep) $(1/2, 0)$, then $\psi^\dagger$ transforms under $(0, 1/2)$ and it can be represented as $\psi^\dagger_{\dot a} = (\psi_a)^\dagger$. This notation is called dotted-undotted and corresponds to right- and left-handed spinors, respectively. We wonder whether a symbol that allows us to raise and lower the spinor indices a and $\dot a$ exists. If so, it must be a Lorentz invariant under the combination of two $(1/2, 0)$ irreps,

$$(1/2, 0) \otimes (1/2, 0) \sim (0, 0) \oplus (1, 0)\,. \tag{I.1}$$

The presence of an antisymmetric singlet $(0, 0)$ reveals the existence of such an object, that we can take as the Levi-Civita symbol ϵ_{ab}. Likewise, for two $(0, 1/2)$ irreps there is also a singlet representation associated to the invariant $\epsilon_{\dot a \dot b}$.

Because ϵ is an invariant, it must hold

$$\epsilon_{ab} = U_L(\Lambda)_a{}^c U_L(\Lambda)_b{}^d \epsilon_{cd}\,, \tag{I.2}$$

where $U_L(\Lambda)$ represents a Lorentz transform on a left-handed spinor. Conversely, with right transformations $U_R(\Lambda)$,

$$\epsilon_{\dot a \dot b} = U_R(\Lambda)_{\dot a}{}^{\dot c} U_R(\Lambda)_{\dot b}{}^{\dot d} \epsilon_{\dot c \dot d}\,. \tag{I.3}$$

We can choose

$$\epsilon_{12} = \epsilon^{12} = \epsilon_{\dot 2 \dot 1} = \epsilon^{\dot 2 \dot 1} = +1\,. \tag{I.4}$$

Note that $\epsilon_a{}^b = -\epsilon^b{}_a = \delta^b_a$ and $\epsilon^{\dot a}{}_{\dot b} = -\epsilon_{\dot b}{}^{\dot a} = \delta^{\dot a}_{\dot b}$, as it is a consequence from considering the rule of raising undotted (dotted) indices by the left (right) and lowering by the right (left):

$$\psi^a = \epsilon^{ab}\psi_b\,, \quad \psi_a = \psi^b \epsilon_{ba}\,, \quad \psi^{\dagger \dot a} = \psi^\dagger_{\dot b}\epsilon^{\dot b \dot a}\,, \quad \psi^\dagger_{\dot a} = \epsilon_{\dot a \dot b}\psi^{\dagger \dot b}\,. \tag{I.5}$$

When two spinors are contracted and indices are omitted, the convention for left-handed fields is that the contraction is made up-down, while for right-handed fields is down-up:

$$\psi\chi \equiv \psi^a\chi_a = \epsilon^{ab}\psi_b\chi_a = -\epsilon^{ba}\psi_b\chi_a = -\psi_b\chi^b = \chi^b\psi_b \Rightarrow \psi\chi = \chi\psi\,, \tag{I.6}$$

$$\psi^\dagger\chi^\dagger \equiv \psi^\dagger_{\dot a}\chi^{\dagger\dot a} = -\psi^{\dagger\dot a}\chi^\dagger_{\dot a} = \chi^\dagger_{\dot a}\psi^{\dagger\dot a} \Rightarrow \psi^\dagger\chi^\dagger = \chi^\dagger\psi^\dagger\,. \tag{I.7}$$

Now, consider a field carrying two indices, dotted and undotted. An object of this class belongs to the irrep $(1/2,0)\otimes(0,1/2)\sim(1/2,1/2)$, which is an operator that belongs to a vector representation. Therefore, there must exist a mapping between the usual vector representation A^μ and its spinor representation, denoted $A_{a\dot a}$. Such a mapping must be provided by an invariant symbol with two (dotted $\dot a$ and undotted a) spinor indices plus a Lorentz index μ to be contracted with that of A^μ. Indeed, that symbol exists since we can find a singlet representation:

$$\underbrace{(1/2,0)}_{a}\otimes\underbrace{(0,1/2)}_{\dot a}\otimes\underbrace{(1/2,1/2)}_{\mu}\sim\underbrace{(0,0)}_{\sigma^\mu_{a\dot a}}\oplus\cdots\,, \tag{I.8}$$

This invariant structure is σ^μ, related to the Dirac-gamma matrices as

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{a\dot a} \\ (\bar\sigma^\mu)^{\dot a a} & 0 \end{pmatrix}\,, \quad (\sigma^\mu)_{a\dot a} = (1,\vec\sigma)\,, \quad (\bar\sigma^\mu)^{\dot a a} = (\sigma^\mu)^{a\dot a} = (1,-\vec\sigma)\,, \tag{I.9}$$

where $\vec\sigma$ is a vector containing the Pauli matrices. The symbols σ^μ and $\bar\sigma^\mu$ satisfy the following properties:

$$(\sigma^\mu)_{a\dot a}(\bar\sigma^\nu)^{\dot a a} = 2g^{\mu\nu}\,, \quad (\sigma^\mu)_{a\dot a}(\bar\sigma_\mu)^{\dot b b} = 2\delta^b_a\delta^{\dot b}_{\dot a}\,, \tag{I.10}$$

and, for being an invariant:

$$(\sigma^\mu)_{a\dot a} = \Lambda^\mu{}_\nu U_L(\Lambda)_a{}^b U_R(\Lambda)_{\dot a}{}^{\dot b}(\sigma^\nu)_{b\dot b}\,, \tag{I.11}$$

This feature provides the appropriate transformation by defining

$$A_{a\dot a}(x) = (\sigma^\mu)_{a\dot a}A_\mu(x) \tag{I.12}$$

as the mapping between usual vector representation A_μ and the spinor representation $A_{a\dot a}$. Therefore, a Lorentz transformation takes the form:

$$A'_{a\dot a}(x') = U_L(\Lambda)_a{}^b U_R(\Lambda)_{\dot a}{}^{\dot b}A_{b\dot b}(x)\,. \tag{I.13}$$

The inverse of the mapping (I.12) is given by relations (I.10) as

$$A^\mu(x) = \frac{1}{2}A_{a\dot a}(x)(\bar\sigma^\mu)^{\dot a a}\,, \tag{I.14}$$

and it also holds

$$A_\mu B^\mu = \frac{1}{2}A_{a\dot a}\bar B^{\dot a a}\,, \tag{I.15}$$

where $\bar B^{\dot a a} = B_\mu(\bar{\sigma^\mu})^{\dot a a}$ has been defined. This matrix is in relation with $B^{a\dot a}$ as follows:

$$B^{a\dot a} = \epsilon^{ab}\epsilon^{\dot b\dot a}B_{b\dot b} = \bar B^{\dot a a}\,, \tag{I.16}$$

which agrees with $(\bar\sigma^\mu)^{\dot a a} = (\sigma^\mu)^{a\dot a}$.

Now that we have the basics, we can apply this formalism to the light-cone coordinates used extensively in this thesis.

Light-cone coordinates

In general, any four-vector v can be decomposed in the so-called *light-cone coordinates*. These coordinates are defined by means of two lightlike vectors n and n' such that $nn' \neq 0$. Thus, v can be expressed as

$$v^\mu = v^+ n'^\mu + v^- n^\mu + v^\mu_\perp \,, \tag{I.17}$$

provided that $v_\perp n = v_\perp n' = 0$.

Now, the idea is to translate this light-cone description into the spinor formalism described above. To do so, one can define two spinors λ and μ such that

$$n_{a\dot{a}} = n^\mu (\sigma_\mu)_{a\dot{a}} = \lambda_a \lambda^\dagger_{\dot{a}} \,, \quad n'_{a\dot{a}} = n'^\mu (\sigma_\mu)_{a\dot{a}} = \mu_a \mu^\dagger_{\dot{a}} \,. \tag{I.18}$$

The lightlike condition on n and n' implies

$$\lambda\lambda = \lambda^\dagger \lambda^\dagger = \mu\mu = \mu^\dagger \mu^\dagger = 0 \,, \tag{I.19}$$

where Eq. (I.15) has been used, and the normalization $nn' \neq 0$ can be expressed in the spinor formalism as

$$2nn' = n_{a\dot{a}} n'^{a\dot{a}} = (\mu\lambda)(\lambda^\dagger \mu^\dagger) \neq 0 \,. \tag{I.20}$$

Hence, considering the two expressions above one can conclude that the different combinations of spinors λ and μ form a basis such that any four-vector v in its spinor representation can be decomposed as

$$2(nn') v_{a\dot{a}} = v^{++} \mu_a \mu^\dagger_{\dot{a}} + v^{--} \lambda_a \lambda^\dagger_{\dot{a}} + v^{-+} \lambda_a \mu^\dagger_{\dot{a}} + v^{+-} \mu_a \lambda^\dagger_{\dot{a}} \,, \tag{I.21}$$

where

$$\begin{aligned} v^{++} &= \lambda^a v_{a\dot{a}} \lambda^{\dagger\,\dot{a}} \,, \quad v^{--} = \mu^a v_{a\dot{a}} \mu^{\dagger\,\dot{a}} \,, \\ v^{-+} &= \mu^a v_{a\dot{a}} \lambda^{\dagger\,\dot{a}} \,, \quad v^{+-} = \lambda^a v_{a\dot{a}} \mu^{\dagger\,\dot{a}} \,. \end{aligned} \tag{I.22}$$

Taking into account that $v_{a\dot{a}} = v_\mu (\sigma^\mu)_{a\dot{a}}$, then

$$v_{a\dot{a}} = (v^+ n'_\mu + v^- n_\mu + v_{\perp,\mu})(\sigma^\mu)_{a\dot{a}} \,, \tag{I.23}$$

and comparing with (I.21), we find:

$$v^+ = \frac{v^{++}}{2(nn')} \,, \quad v^- = \frac{v^{--}}{2(nn')} \,, \quad v_{\perp,a\dot{a}} = v_{\perp,\mu} (\sigma^\mu)_{a\dot{a}} = \frac{v^{-+} \lambda_a \mu^\dagger_{\dot{a}} + v^{+-} \mu_a \lambda^\dagger_{\dot{a}}}{2(nn')} \,. \tag{I.24}$$

In fact, the transverse metric in spinor formalism is given by

$$\begin{aligned} g_{\perp,ab\dot{a}\dot{b}} &= g_{\perp,\mu\nu} (\sigma^\mu)_{a\dot{a}} (\sigma^\nu)_{b\dot{b}} \\ &= \left(g_{\mu\nu} - \frac{n_\mu n'_\nu + n_\nu n'_\mu}{nn'} \right) (\sigma^\mu)_{a\dot{a}} (\sigma^\nu)_{b\dot{b}} \end{aligned}$$

$$= -2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} - \frac{1}{nn'}(\lambda_a\lambda^\dagger_{\dot{a}}\mu_b\mu^\dagger_{\dot{b}} + \lambda_b\lambda^\dagger_{\dot{b}}\mu_a\mu^\dagger_{\dot{a}})\,, \tag{I.25}$$

where Eqs. (I.10) and (I.18) were used. It projects the transverse component in spinor formalism:

$$\begin{aligned} v_{\perp,a\dot{a}} &= g_{\perp,\mu\nu}v^\nu(\sigma^\mu)_{a\dot{a}} \\ &= \frac{1}{2}g_{\perp,\mu\nu}v_{b\dot{b}}(\sigma^\nu)^{b\dot{b}}(\sigma^\mu)_{a\dot{a}} \\ &= \frac{1}{2}v_{b\dot{b}}g_{\perp,aB\dot{a}\dot{B}}\epsilon^{bB}\epsilon^{\dot{B}\dot{b}} \\ &= \frac{1}{2(nn')}(v^{+-}\mu_a\lambda^\dagger_{\dot{a}} + v^{-+}\lambda_a\mu^\dagger_{\dot{a}})\,. \end{aligned} \tag{I.26}$$

In the last step, the spinor representations (I.21) and (I.25) were introduced.

The last spinor representation that we need is that of the transverse Levi-Civita tensor

$$\epsilon^{\mu\nu}_\perp = \frac{1}{nn'}\epsilon^{\mu\nu\rho\sigma}n'_\rho n_\sigma\,, \tag{I.27}$$

which appears in the Compton tensor of a spin-1/2 targets, but it is also required to build a basis in the transverse plane. Given a vector v whose perpendicular components are given as $v_\perp$, there is an independent (dual) vector $\widetilde{v}$ whose trasverse components are given in vector representation by:

$$\widetilde{v}^\mu_\perp = \epsilon^{\mu\nu}_\perp v_\nu\,, \tag{I.28}$$

while its spinor representation is:

$$\widetilde{v}_{\perp,a\dot{a}} = \frac{1}{2}v^{b\dot{b}}_\perp\epsilon_{\perp,ab\dot{a}\dot{b}}\,, \tag{I.29}$$

where

$$\epsilon_{\perp,ab\dot{a}\dot{b}} = \epsilon^{\mu\nu}_\perp(\sigma_\mu)_{a\dot{a}}(\sigma_\nu)_{b\dot{b}}\,. \tag{I.30}$$

To unveil the form of this representation, we expand it by means of the spinors λ, μ

$$\epsilon_{\perp,ab\dot{a}\dot{b}} = A_1\lambda_a\lambda^\dagger_{\dot{b}}\mu_b\mu^\dagger_{\dot{a}} + A_2\lambda_b\lambda^\dagger_{\dot{a}}\mu_a\mu^\dagger_{\dot{b}} + A_3\lambda_a\lambda^\dagger_{\dot{a}}\mu_b\mu^\dagger_{\dot{b}} + A_4\lambda_b\lambda^\dagger_{\dot{b}}\mu_a\mu^\dagger_{\dot{a}} \tag{I.31}$$

and apply its orthogonality with n and n'. This provides the conditions:

$$A_3 = A_4 = 0\,. \tag{I.32}$$

We also notice that because of

$$\epsilon_{\perp,ab\dot{a}\dot{b}} = -\epsilon_{\perp,ba\dot{b}\dot{a}}\,, \tag{I.33}$$

then

$$A_1 = -A_2\,. \tag{I.34}$$

All in all, we still need to determine A_1. To do so, we take into account that

$$\widetilde{\widetilde{v}}^\lambda_\perp = \epsilon^{\lambda\mu}_\perp\epsilon_{\perp,\mu\nu}v^\nu_\perp = -v^\lambda_\perp\,, \tag{I.35}$$

whose spinor representation leads to:

$$\widetilde{\widetilde{v}}_{\perp,a\dot{a}} = \frac{A_1^2}{4}(2nn')^2 v_{\perp,a\dot{a}} \Rightarrow \frac{A_1^2}{4}(2nn')^2 = -1 \Rightarrow A_1 = \frac{\pm i}{nn'} \,. \tag{I.36}$$

For simplicity, we can choose the plus sign [137], hence

$$\epsilon_{\perp,ab\dot{a}\dot{b}} = \frac{i}{nn'}(\lambda_a \lambda^\dagger_{\dot{b}} \mu_b \mu^\dagger_{\dot{a}} - \lambda_b \lambda^\dagger_{\dot{a}} \mu_a \mu^\dagger_{\dot{b}}) \,. \tag{I.37}$$

Finally, with this form of the transverse Levi-Civita, the dual $\widetilde{v}_\perp$ reads:

$$\widetilde{v}_{\perp,a\dot{a}} = \frac{i}{2(nn')}(p^{+-}\mu_a \lambda^\dagger_{\dot{a}} - p^{-+}\lambda_a \mu^\dagger_{\dot{a}}) \,. \tag{I.38}$$

J

Fourier transforms

For the kinematic-twist expansion of the Compton tensor we require the following Fourier transform:

$$I_n^{(D)} = i \int d^D z \, e^{irz} \frac{1}{(-z^2 + i0)^n} \,, \tag{J.1}$$

$$\hat{I}_n^{(D),\,\mu} = i \int d^D z \, e^{irz} \frac{z^\mu}{(-z^2 + i0)^n} \,, \tag{J.2}$$

$$\widetilde{I}_n^{(D),\,\mu\nu} = i \int d^D z \, e^{irz} \frac{z^\mu z^\nu}{(-z^2 + i0)^n} \,, \tag{J.3}$$

$$\check{I}_n^{(D),\,\mu\nu\alpha} = i \int d^D z \, e^{irz} \frac{z^\mu z^\nu z^\alpha}{(-z^2 + i0)^n} \,, \tag{J.4}$$

$$\bar{I}_n^{(D),\,\mu\nu\alpha\beta} = i \int d^D z \, e^{irz} \frac{z^\mu z^\nu z^\alpha z^\beta}{(-z^2 + i0)^n} \,, \tag{J.5}$$

where n is a positive integer. For $n \in \{1, 2, 3\}$, the relevant integrals take the form:

$$I_1^{(4)} = \frac{-4\pi^2}{r^2 + i0} \,, \tag{J.6}$$

$$I_2^{(4+2\varepsilon)} = \frac{\pi^2}{\mu^{2\varepsilon}} \left(\frac{1}{\varepsilon} - \ln \left(\frac{r^2 + i0}{-\mu^2} \right) + \ln(4\pi) - \gamma_E \right) + O(\varepsilon) \,, \tag{J.7}$$

$$\hat{I}_1^{(4),\,\mu} = -i \frac{8\pi^2 r^\mu}{(r^2 + i0)^2} \,, \tag{J.8}$$

$$\hat{I}_2^{(4),\,\mu} = i \frac{2\pi^2 r^\mu}{r^2 + i0} \,, \tag{J.9}$$

$$\widetilde{I}_1^{(4),\,\mu\nu} = -\frac{8\pi^2}{(r^2 + i0)^2} \left[g^{\mu\nu} - \frac{4 r^\mu r^\nu}{r^2 + i0} \right] \,, \tag{J.10}$$

$$\widetilde{I}_2^{(4),\,\mu\nu} = \frac{2\pi^2}{r^2+i0}\left[g^{\mu\nu} - \frac{2r^\mu r^\nu}{r^2+i0}\right], \tag{J.11}$$

$$\widetilde{I}_3^{(4+2\varepsilon),\,\mu\nu} = \frac{\pi^2}{4\mu^{2\varepsilon}}\left[-g^{\mu\nu}\left(\frac{1}{\varepsilon} + \ln\left(\frac{-\mu^2}{r^2+i0}\right) - \gamma_E + O(\varepsilon)\right) + 2r^\mu r^\nu \frac{1}{r^2+i0}\right], \tag{J.12}$$

$$\check{I}_1^{(4)\,,\mu\nu\alpha} = -i\frac{32\pi^2}{(r^2+i0)^3}\left[r^\alpha g^{\mu\nu} + r^\mu g^{\nu\alpha} + r^\nu g^{\mu\alpha} - \frac{6r^\mu r^\nu r^\alpha}{r^2+i0}\right], \tag{J.13}$$

$$\check{I}_2^{(4),\,\mu\nu\alpha} = i\frac{4\pi^2}{(r^2+i0)^2}\left[r^\alpha g^{\mu\nu} + r^\mu g^{\nu\alpha} + r^\nu g^{\mu\alpha} - \frac{4r^\mu r^\nu r^\alpha}{r^2+i0}\right], \tag{J.14}$$

$$\check{I}_3^{(4),\,\mu\nu\alpha} = -i\frac{\pi^2}{2(r^2+i0)}\left[r^\alpha g^{\mu\nu} + r^\mu g^{\nu\alpha} + r^\nu g^{\mu\alpha} - \frac{2r^\mu r^\nu r^\alpha}{r^2+i0}\right], \tag{J.15}$$

$$\begin{aligned}\bar{I}_1^{(4),\,\mu\nu\alpha\beta} = &\frac{192\pi^2}{(r^2+i0)^4}[r^\alpha r^\beta g^{\mu\nu} + r^\mu r^\nu g^{\alpha\beta} + 2r^\mu r^{(\alpha|}g^{\nu|\beta)} + 2r^\nu r^{(\alpha|}g^{\mu|\beta)}] \\ &- \frac{32\pi^2}{(r^2+i0)^3}[g^{\alpha\beta}g^{\mu\nu} + 2g^{\mu(\alpha|}g^{\nu|\beta)}] - \frac{1536\pi^2}{(r^2+i0)^5}r^\mu r^\nu r^\alpha r^\beta,\end{aligned} \tag{J.16}$$

$$\begin{aligned}\bar{I}_2^{(4),\,\mu\nu\alpha\beta} = &\frac{-16\pi^2}{(r^2+i0)^3}[r^\alpha r^\beta g^{\mu\nu} + r^\mu r^\nu g^{\alpha\beta} + 2r^\mu r^{(\alpha|}g^{\nu|\beta)} + 2r^\nu r^{(\alpha|}g^{\mu|\beta)}] \\ &+ \frac{4\pi^2}{(r^2+i0)^2}[g^{\alpha\beta}g^{\mu\nu} + 2g^{\mu(\alpha|}g^{\nu|\beta)}] + \frac{96\pi^2}{(r^2+i0)^4}r^\mu r^\nu r^\alpha r^\beta,\end{aligned} \tag{J.17}$$

$$\begin{aligned}\bar{I}_3^{(4),\,\mu\nu\alpha\beta} = &\frac{\pi^2}{(r^2+i0)^2}[r^\alpha r^\beta g^{\mu\nu} + r^\mu r^\nu g^{\alpha\beta} + 2r^\mu r^{(\alpha|}g^{\nu|\beta)} + 2r^\nu r^{(\alpha|}g^{\mu|\beta)}] \\ &- \frac{\pi^2}{2(r^2+i0)}[g^{\alpha\beta}g^{\mu\nu} + 2g^{\mu(\alpha|}g^{\nu|\beta)}] - \frac{4\pi^2}{(r^2+i0)^3}r^\mu r^\nu r^\alpha r^\beta,\end{aligned} \tag{J.18}$$

where γ_E is the Euler-Mascheroni constant and $(\mu_1 \cdots \mu_n)$ stands for symmetrization and normalization by $n!$.

The integrals labelled as (J.1) can be calculated using the Schwinger representation for the rational function:

$$I_n^{(D)} = \frac{1}{\Gamma(n)}\int d^Dz\, e^{irz}\int_0^\infty da\, a^{n-1}e^{-a(z^2+i0)}, \tag{J.19}$$

where z and r have been changed to Euclidean spacetime: $z^0|_{\text{Euclidean}} = -iz^0|_{\text{Minkowski}}$ and $r^0|_{\text{Euclidean}} = ir^0|_{\text{Minkowski}}$.

With $b = a^{-1}$,

$$\begin{aligned} I_n^{(D)} &= \frac{\pi^{D/2}}{\Gamma(n)} \int_0^\infty db\, b^{(2-n+\varepsilon)-1} \exp\{-(r^2/4)b - i0/b\} \\ &= \frac{\pi^{D/2}}{\Gamma(n)} \int_0^\infty db\, b^{(2-n+\varepsilon)-1} \exp\left\{-\frac{r^2 - i0}{4} b\right\}. \end{aligned} \tag{J.20}$$

This integral can be related to the $\Gamma(2 - n + \varepsilon)$ function as long as the argument has a strictly positive real part. Hence, going back to Minkowski's spacetime we conclude:

$$I_n^{(D)} = \frac{\pi^{D/2}}{\Gamma(n)} \Gamma(D/2 - n) \left(\frac{-4}{r^2 + i0}\right)^{D/2-n}, \text{ for } 0 \leq \mathfrak{Re}(n) \leq 2\,. \tag{J.21}$$

Particularizing for $n = 1$

$$I_1^{(4)} = \frac{-4\pi^2}{r^2 + i0} \tag{J.22}$$

and for $n = 2$ with $D = 4 + 2\varepsilon$, $\varepsilon \to 0$,

$$I_2^{(4+2\varepsilon)} = \frac{\pi^2}{\mu^{2\varepsilon}} \left(\frac{1}{\varepsilon} - \ln\left(\frac{r^2 + i0}{-\mu^2}\right) + \ln(4\pi) - \gamma_E\right) + O(\varepsilon)\,. \tag{J.23}$$

Here, we introduced the renormalization scale μ and the Euler-Mascheroni constant γ_E.

The second integral in Eq. (J.3), $\widetilde{I}_n^{(D),\mu\nu}$, can be formulated via derivatives of $I_n^{(D)}$:

$$\widetilde{I}_n^{(D),\mu\nu} = -\frac{\partial}{\partial r_\mu}\frac{\partial}{\partial r_\nu} I_n^{(D)}\,. \tag{J.24}$$

In particular, for $n = 1$ we have

$$\widetilde{I}_1^{(4),\mu\nu} = -\frac{8\pi^2}{(r^2 + i0)^2}\left[g^{\mu\nu} - \frac{4r^\mu r^\nu}{r^2 + i0}\right], \tag{J.25}$$

and for $n = 2$

$$\widetilde{I}_2^{(4),\mu\nu} = \frac{2\pi^2}{r^2 + i0}\left[g^{\mu\nu} - \frac{2r^\mu r^\nu}{r^2 + i0}\right]. \tag{J.26}$$

For the case $n = 3$, $I_3^{(4+2\varepsilon)}$ cannot be computed using Eq. (J.21). Hence, in order to obtain $\widetilde{I}_3^{(4+2\varepsilon),\mu\nu}$ we need to go back to the integral expression of $I_3^{(4+2\varepsilon)}$:

$$\begin{aligned} \widetilde{I}_3^{(4+2\varepsilon),\mu\nu} &= -\frac{\partial}{\partial r_\mu}\frac{\partial}{\partial r_\nu} I_3^{(4+2\varepsilon)} \\ &= -\frac{1}{2}\frac{\pi^{D/2}}{\Gamma(3)} \int_0^\infty db\, b^{\varepsilon-1} \left[g^{\mu\nu} + \frac{1}{2} r^\mu r^\nu b\right] \exp\left\{-\frac{r^2 - i0}{4} b\right\} \\ &= -\frac{1}{2}\frac{\pi^{D/2}}{\Gamma(3)} \left[g^{\mu\nu}\Gamma(\varepsilon)\left(\frac{-4}{r^2 + i0}\right)^{\varepsilon} + \frac{1}{2} r^\mu r^\nu \Gamma(\varepsilon + 1)\left(\frac{-4}{r^2 + i0}\right)^{\varepsilon+1}\right] \\ &= \frac{\pi^2}{4\mu^{2\varepsilon}}\left[-g^{\mu\nu}\left(\frac{1}{\varepsilon} + \ln\left(\frac{-\mu^2}{r^2 + i0}\right) - \gamma_E + O(\varepsilon)\right) + 2r^\mu r^\nu \frac{1}{r^2 + i0}\right]. \end{aligned} \tag{J.27}$$

In the first line, r in derivatives is expressed in Minkowsky spacetime while in the second line, after performing the derivation, we returned r to Euclidean spacetime.

This change is needed in order to have a convergent integrand (exponential that decays as $b \to \infty$) and so to relate the integral over b to the Γ function.

Similarly, the vectors $\hat{I}_n$ and the 3- and 4-ranked tensors $\check{I}_n, \bar{I}_n$ can be related to derivatives of the scalars I_n and 2-ranked tensors $\widetilde{I}_n$ with respect to the auxiliary vector r:

$$\hat{I}_n^{(D),\mu} = -i\frac{\partial}{\partial r_\mu} I_n^{(D)}, \tag{J.28}$$

$$\check{I}_n^{(D),\mu\nu\alpha} = -i\frac{\partial}{\partial r_\alpha} \widetilde{I}_n^{(D),\mu\nu}. \tag{J.29}$$

$$\bar{I}_n^{(D),\mu\nu\alpha\beta} = -i\frac{\partial}{\partial r_\beta} \check{I}_n^{(D),\mu\nu\alpha}. \tag{J.30}$$

K

Prescription to map DDs to GPDs in convolutions

Up to kinematic twist-4, the hard-coefficient functions are given by the integrals

$$\mathbb{I}_0[g] = \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi)g(x,\xi,\rho)\,, \tag{K.1}$$

$$\mathbb{I}_k[f] = \int_{-1}^{1} dx \iint_{\mathbb{D}} d\beta d\alpha\, 2\Phi^{(+)}\delta(x-\beta-\alpha\xi)\frac{(\xi\beta)^k}{\xi^2} f(x/\xi,\rho/\xi)\,, \quad k>0 \tag{K.2}$$

For our purposes, it is enough to solve $\mathbb{I}_0$, $\mathbb{I}_1$ and $\mathbb{I}_2$. Functional $\mathbb{I}_0$ makes use of relation (3.192) and integration by parts. The result, for any function $g = g(x,\xi,\rho)$, is:

$$\mathbb{I}_0[g] = -\int_{-1}^{1} dx\ (\partial_x g)\,\frac{H^{(+)}}{2}\,. \tag{K.3}$$

For $k \in \{1,2\}$, we have the following solutions:

$$\mathbb{I}_1[f] = \int_{-1}^{1} dx\ \left[\partial_\xi\left(\frac{f}{2}H^{(+)}\right) - \frac{f}{2\xi}H^{(+)}\right]\,, \tag{K.4}$$

$$\mathbb{I}_2[f] = \xi^3\partial_\xi^2 \int_{-1}^{1} dx\ \frac{\mathcal{Y}[f]H^{(+)}}{2\xi}\,, \tag{K.5}$$

where $f = f(x/\xi,\rho/\xi)$ and

$$\mathcal{Y}[f] = \int_x^1 dx'\ f(x'/\xi,\rho/\xi)\,. \tag{K.6}$$

For $k=1$, note that

$$\beta\delta(x-\beta-\alpha\xi) = (x\partial_x + \xi\partial_\xi)\theta(x-\beta-\alpha\xi)\,, \tag{K.7}$$

which together with

$$\iint_{\mathbb{D}} d\beta d\alpha\ \theta(x-\beta-\alpha\xi)\Phi^{(+)}(\beta,\alpha,t) = \frac{1}{4}H^{(+)}(x,\xi,t)\,, \tag{K.8}$$

allows to write

$$\mathbb{I}_1[f] = \int_{-1}^{1} dx\, \frac{f}{\xi}(x\partial_x + \xi\partial_\xi)\frac{H^{(+)}(x,\xi,t)}{2}\,. \tag{K.9}$$

Integration by parts plus the relation

$$\partial_\xi f(x/\xi, \rho/\xi) = -\frac{x}{\xi}\partial_x f(x/\xi, \rho/\xi)\,, \tag{K.10}$$

which can be used to trade ∂_ξ by ∂_x in order to solve $\int dx$, renders

$$\mathbb{I}_1[f] = \int_{-1}^{1} dx\, \left[\partial_\xi \left(\frac{f}{2}H^{(+)}\right) - \frac{f}{2\xi}H^{(+)}\right]\,. \tag{K.11}$$

For $k = 2$, note that

$$\beta^2\delta(x-\beta-\alpha\xi) = (x^2\partial_x + 2x\xi\partial_\xi)\theta(x-\beta-\alpha\xi) + \xi^2\partial_\xi^2 \int_{-1}^{x} dx'\, \theta(x'-\beta-\alpha\xi)\,, \tag{K.12}$$

and that we can exchange the order of integration over x and x' variables via

$$\begin{aligned}\int_{-1}^{1} dx \int_{-1}^{x} dx'\, C(x)H^{(+)}(x',\xi,t) &= \int_{-1}^{1} dx' \int_{x'}^{1} dx\, C(x)H^{(+)}(x',\xi,t)\\ &= \int_{-1}^{1} dx\, H^{(+)}(x,\xi,t) \int_{x}^{1} dx'\, C(x')\,, \end{aligned} \tag{K.13}$$

for any function $C(x)$. After integration by parts, these formulas together with Eqs. (K.8) and (K.10) yield:

$$\mathbb{I}_2[f] = \xi^3\partial_\xi^2 \int_{-1}^{1} dx\, \frac{H^{(+)}(x,\xi,t)}{2\xi} \int_{x}^{1} dx'\, f(x'/\xi, \rho/\xi) \tag{K.14}$$

$$= \xi^3\partial_\xi^2 \int_{-1}^{1} dx\, \frac{\mathcal{Y}[f]H^{(+)}}{2\xi}\,, \tag{K.15}$$

where we identified $\mathcal{Y}[f]$ as in Eq. (K.6).